Learning Constitutive Models in Multiscale Modeling of Materials

4TU.HTM WORKSHOP 2021, 23-06-2021

Ondřej Rokoš, Department of Mechanical Engineering, Eindhoven University of Technology
Theron Guo, Department of Mathematics and Computer Science, Eindhoven University of Technology
Karen Veroy, Department of Mathematics and Computer Science, Eindhoven University of Technology

European Research Council
Established by the European Commission

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Outline

- Problem Statement
- Methodology
- Numerical Examples
- Conclusion & Outlook

Acknowledgments: This result is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Programme (Grant Agreement No. 818473).

Problem Statement

Macroscopic Problem: Find $\bar{\boldsymbol{u}}(\bar{\boldsymbol{X}})$

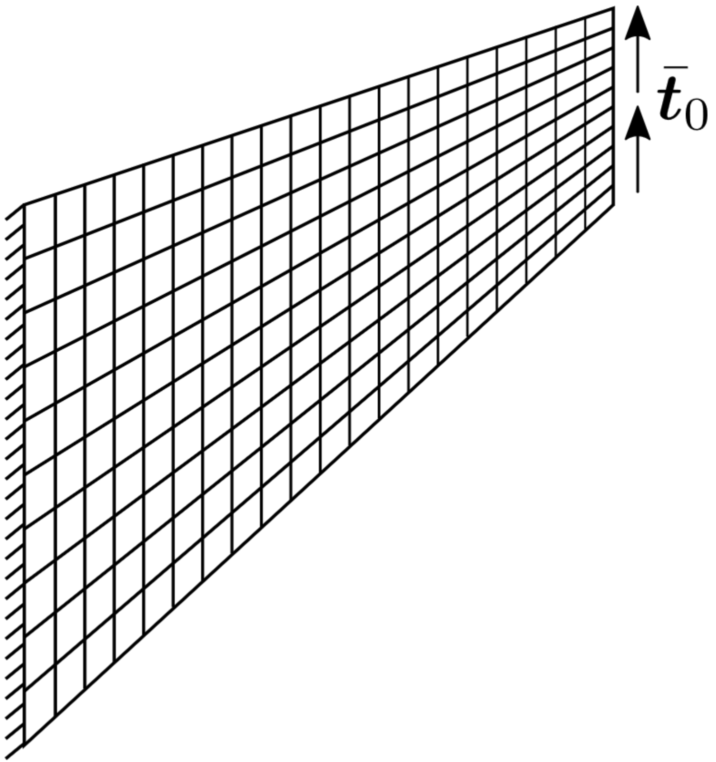
$$\begin{aligned} \mathrm{Div}\,\bar{\boldsymbol{P}}(\bar{\boldsymbol{u}}) &= \boldsymbol{0} && \text{on } \bar{\Omega}, \\ \bar{\boldsymbol{P}}\bar{\boldsymbol{N}} &= \bar{\boldsymbol{t}}_0 && \text{on } \partial\bar{\Omega}^N, \\ \bar{\boldsymbol{u}} &= \bar{\boldsymbol{u}}_0 && \text{on } \partial\bar{\Omega}^D, \end{aligned}$$

Constitutive Model

$$\bar{\boldsymbol{P}} = \bar{\boldsymbol{P}}(\bar{\boldsymbol{F}}, \boldsymbol{\mu})$$

$$\bar{\boldsymbol{F}} = \mathbf{I} + \frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{\boldsymbol{X}}}$$

$$\bar{\boldsymbol{A}} = \frac{\partial \bar{\boldsymbol{P}}(\bar{\boldsymbol{F}}, \boldsymbol{\mu})}{\partial \bar{\boldsymbol{F}}}$$



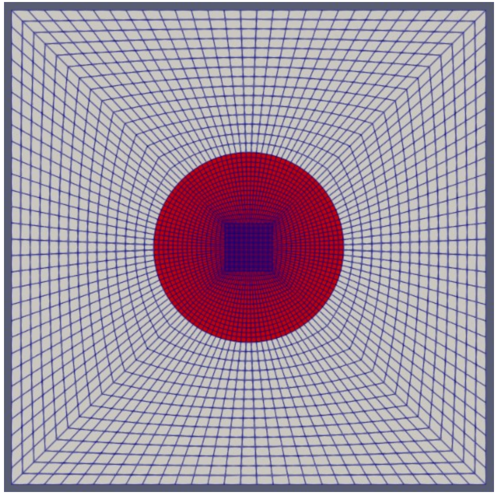
[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Problem Statement

Microscopic Problem: Find $\boldsymbol{u}(\boldsymbol{X})$

$$\mathrm{Div}\boldsymbol{P}(\boldsymbol{u}) = \boldsymbol{0} \quad \text{on } \Omega$$

$$\boldsymbol{u} = \underbrace{(\bar{\boldsymbol{F}} - \mathbf{I})\boldsymbol{X}}_{\text{mean}} + \underbrace{\boldsymbol{w}(\boldsymbol{X})}_{\text{fluctuation}}$$



Neo-Hookean material

$$W(\boldsymbol{F}, \boldsymbol{\mu}) = C_1(\mathrm{Tr}(\boldsymbol{F}^T\boldsymbol{F}) - 3 - 2\ln(\det\boldsymbol{F})) + D_1(\det\boldsymbol{F} - 1)^2$$

$$\boldsymbol{P}(\boldsymbol{F}, \boldsymbol{\mu}) = \frac{\partial W}{\partial \boldsymbol{F}}$$

$$\boldsymbol{A}(\boldsymbol{F}, \boldsymbol{\mu}) = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}}$$

$\bar{\boldsymbol{F}}$ loading: macroscopic deformation gradient

$\boldsymbol{\mu}$ parameters: material & geometry

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Problem Statement

Scale Coupling $\langle \bullet \rangle = |\Omega|^{-1} \int_{\Omega} \bullet \, dV$

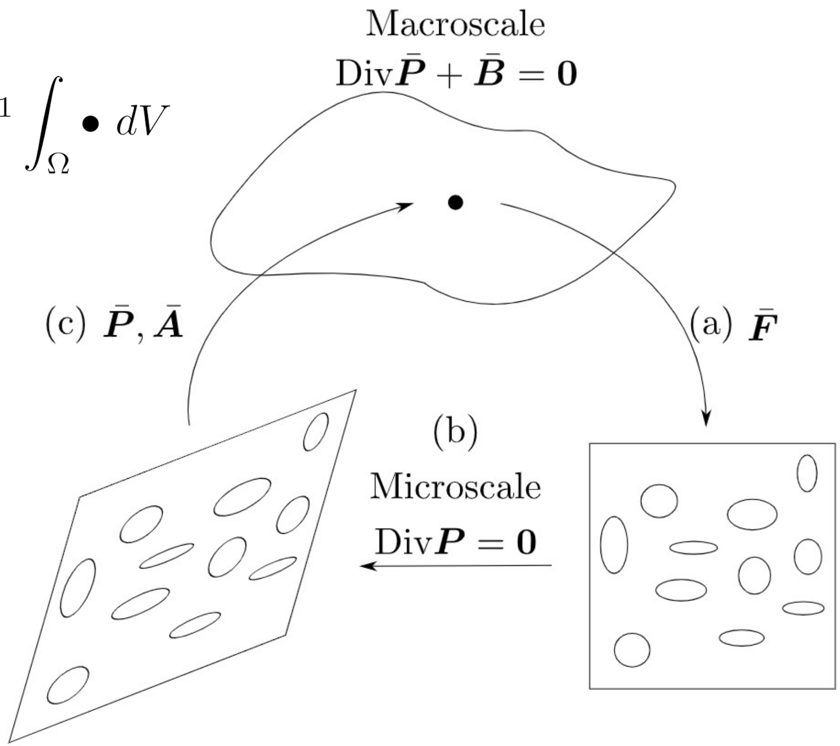
Hill-Mandel Condition

$$\langle \boldsymbol{P} : \delta \boldsymbol{F} \rangle = \bar{\boldsymbol{P}} : \delta \bar{\boldsymbol{F}}$$

$$\Rightarrow \begin{array}{lll} \boldsymbol{w} = 0 & \text{on } \partial\Omega & \text{(KBC)} \\ \boldsymbol{w} \text{ periodic} & \text{on } \partial\Omega & \text{(PBC)} \end{array}$$

Effective Quantities

$$\bar{\boldsymbol{P}} = \langle \boldsymbol{P} \rangle$$
$$\bar{\boldsymbol{A}} \neq \langle \boldsymbol{A} \rangle$$



[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Problem Statement

- Full two-scale simulations → **too expensive**
  (multi-query contexts: optimization, materials design, etc.)
- **Non-intrusive and accurate approximation of the microscopic stress field for different parameters**
- Obtain rapidly **effective stress** → reducing to a **single-scale** problem
-

$$\boldsymbol{P}(\boldsymbol{X}, \bar{\boldsymbol{F}}, \boldsymbol{\mu}), \quad \bar{\boldsymbol{P}}(\bar{\boldsymbol{F}}, \boldsymbol{\mu})$$

- **Accurate effective stiffness and sensitivities**

$$\bar{\boldsymbol{A}} = \frac{\partial \bar{\boldsymbol{P}}(\bar{\boldsymbol{F}}, \boldsymbol{\mu})}{\partial \bar{\boldsymbol{F}}}, \quad \frac{\partial \bar{\boldsymbol{P}}(\bar{\boldsymbol{F}}, \boldsymbol{\mu})}{\partial \boldsymbol{\mu}}$$

Methodology

Intrusive RB Method

[Yvonnet, J. and He, Q.C., 2007, JCP 223]
[Radermacher, A., et al., 2016, AMSES 3(1)]
[Hernández, J. A., et al., 2014, CMAME 276]
[Soldner, Dominic, et al., 2017, Comput. Mech. 60(4)]

Approximate displacement field

$$\boldsymbol{u}(\boldsymbol{X}, \bar{\boldsymbol{F}}, \boldsymbol{\mu}) \approx \sum_{l=1}^{L} \alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu}) \boldsymbol{\Phi}_l(\boldsymbol{X}) = \boldsymbol{\Phi}\boldsymbol{\alpha}$$ with $\boldsymbol{\Phi}(\boldsymbol{X})$ RB basis functions

Reduced problem

$$\boldsymbol{f}(\boldsymbol{\Phi}\boldsymbol{\alpha}) = \mathbf{0}$$

Need hyperreduction!

$$\boldsymbol{\Phi}^T \left. D\boldsymbol{f}\right|_{\boldsymbol{\Phi}\boldsymbol{\alpha}^k} \boldsymbol{\Phi}\Delta\boldsymbol{\alpha} = -\boldsymbol{\Phi}^T \boldsymbol{f}(\boldsymbol{\Phi}\boldsymbol{\alpha}^k)$$

$$\boldsymbol{\alpha}^{k+1} = \boldsymbol{\alpha}^k + \Delta\boldsymbol{\alpha}$$

- **Hyperreduction** (convergence problems)
- **Intrusive** (requires access to the microscopic solver)

TU/e

Methodology

Non-Intrusive RB Method

[Guo, M. and Hesthaven, J.S., 2018, CMAME 341]
[Kast, M., Guo, M. and Hesthaven, J.S., 2020, CMAME 364]
[Swischuk, Renee, et al., 2019, Comput Fluids 179]

Approximate displacement field

$$\boldsymbol{u}(\boldsymbol{X},\bar{\boldsymbol{F}},\boldsymbol{\mu}) \approx \sum_{l=1}^{L} \alpha_l(\bar{\boldsymbol{F}},\boldsymbol{\mu})\boldsymbol{\Phi}_l(\boldsymbol{X}) = \boldsymbol{\Phi}\boldsymbol{\alpha}$$ with $\boldsymbol{\Phi}(\boldsymbol{X})$ RB basis functions

Instead of solving reduced system, perform a regression to obtain

$$\hat{\alpha}_l : \mathcal{P} \to \mathbb{R} : (\bar{\boldsymbol{F}},\boldsymbol{\mu}) \mapsto \hat{\alpha}_l(\bar{\boldsymbol{F}},\boldsymbol{\mu})$$

For a new pair $(\bar{\boldsymbol{F}},\boldsymbol{\mu})$, the coefficients $\boldsymbol{\alpha}$ can be directly obtained

Methodology

Non-Intrusive RB Method

[Guo, M. and Hesthaven, J.S., 2018, CMAME 341]
[Kast, M., Guo, M. and Hesthaven, J.S., 2020, CMAME 364]
[Swischuk, Renee, et al., 2019, Comput Fluids 179]

Approximate displacement field

$$\boldsymbol{u}(\boldsymbol{X}, \bar{\boldsymbol{F}}, \boldsymbol{\mu}) \approx \sum_{l=1}^{L} \alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu}) \boldsymbol{\Phi}_l(\boldsymbol{X}) = \boldsymbol{\Phi}\boldsymbol{\alpha}$$ with $\boldsymbol{\Phi}(\boldsymbol{X})$ RB basis functions

To obtain stresses, gradient of $\boldsymbol{u}$ is needed, and evaluation of the constitutive model

- **Intrusive**: Need to implement material model
- **Slow**: Need to evaluate material model N_{qp} times

Idea: Approximate stress field directly!

$$\boldsymbol{P}(\boldsymbol{X}, \bar{\boldsymbol{F}}, \boldsymbol{\mu}) \approx \sum_{l=1}^{L} \alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu}) \boldsymbol{B}_l(\boldsymbol{X})$$

Methodology

Proper Orthogonal Decomposition (POD)

1. Collect stress snapshots $\mathbf{S} = \left[\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \dots, \mathbf{P}^{(N_{\text{pod}})}\right]$
2. Compute correlation matrix

$$\mathrm{C}_{ij} = \left(\boldsymbol{P}^{(i)}, \boldsymbol{P}^{(j)}\right)_{L^2(\Omega)} = \int_\Omega \boldsymbol{P}^{(i)} : \boldsymbol{P}^{(j)} dV$$

3. Compute the eigenvalues λ_l and eigenvectors $\mathbf{v}_l$ of $\mathbf{C}$
4. Compute the basis functions

$$\mathbf{B}_l = \frac{1}{\sqrt{\lambda_l}} \mathbf{S} \mathbf{v}_l \qquad (\boldsymbol{B}_i, \boldsymbol{B}_j)_{L^2(\Omega)} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\Rightarrow \boldsymbol{P}^{\mathrm{RB}}(\boldsymbol{X}, \bar{\boldsymbol{F}}, \boldsymbol{\mu}) = \sum_{l=1}^{L} \alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu}) \boldsymbol{B}_l(\boldsymbol{X})$$

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Methodology

$\int_\Omega \boldsymbol{B}_l(\boldsymbol{X})dV$ can be computed offline!

Effective Quantities

$$\bar{\boldsymbol{P}} = \langle \boldsymbol{P}^{\mathrm{RB}} \rangle = |\Omega|^{-1} \int_\Omega \boldsymbol{P}^{\mathrm{RB}} dV = |\Omega|^{-1} \sum_{l=1}^{L} \alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu}) \int_\Omega \boldsymbol{B}_l(\boldsymbol{X}) dV$$

$$\bar{\boldsymbol{A}} = \frac{\partial \bar{\boldsymbol{P}}}{\partial \bar{\boldsymbol{F}}} = |\Omega|^{-1} \sum_{l=1}^{L} \int_\Omega \boldsymbol{B}_l(\boldsymbol{X}) dV \otimes \frac{\partial \alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu})}{\partial \bar{\boldsymbol{F}}}$$

Sensitivities

$$\frac{\partial \bar{\boldsymbol{P}}}{\partial \boldsymbol{\mu}} = |\Omega|^{-1} \sum_{l=1}^{L} \int_\Omega \boldsymbol{B}_l(\boldsymbol{X}) dV \otimes \frac{\partial \alpha_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu})}{\partial \boldsymbol{\mu}}$$

$\Rightarrow$ Construct $\hat{\alpha}_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu}), \dfrac{\partial \hat{\alpha}_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu})}{\partial \bar{\boldsymbol{F}}}, \dfrac{\partial \hat{\alpha}_l(\bar{\boldsymbol{F}}, \boldsymbol{\mu})}{\partial \boldsymbol{\mu}}$ with Gaussian Process Regression!

Methodology

[Guo, M. and Hesthaven, J.S., 2018, CMAME 341]
[Rasmussen, C.E., 2003, *Summer school on machine learning*]

Gaussian Process Regression (GPR)

Given some data $\{\mathbf{X}^{(i)}, y^{(i)}\}_{i=1}^{N}$, we want to find a scalar regression function which is distributed as a Gaussian Process with zero mean function and kernel $k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}')$

$$f \sim \mathcal{GP}(0, k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}'))$$

Automatic relevance determination (ARD) squared exponential kernel

$$k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}') = \sigma_f^2 \exp\left(-\frac{1}{2}\sum_{k=1}^{d}\frac{(X_k - X'_k)^2}{l_k^2}\right)$$

Methodology

[Guo, M. and Hesthaven, J.S., 2018, CMAME 341]
[Rasmussen, C.E., 2003, *Summer school on machine learning*]

Gaussian Process Regression (GPR)

Given some data $\{\mathbf{X}^{(i)}, y^{(i)}\}_{i=1}^{N}$, we want to find a scalar regression function which is distributed as a Gaussian Process with zero mean function and kernel $k_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}')$

$$f^*|(\mathbf{X}, \mathbf{y}) \sim \mathcal{GP}(m^*, k^*),$$

$$m^*(\mathbf{x}) = \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{x})^T \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X})^{-1} \mathbf{y}(\mathbf{X}),$$

$$k^*(\mathbf{x}, \mathbf{x}') = k_{\boldsymbol{\theta}}(\mathbf{x}, \mathbf{x}') - \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{x})^T \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X})^{-1} \mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{x}')$$

where $\mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{x}) = [k_{\boldsymbol{\theta}}(\mathbf{X}^{(1)}, \mathbf{x}), k_{\boldsymbol{\theta}}(\mathbf{X}^{(2)}, \mathbf{x}), \ldots, k_{\boldsymbol{\theta}}(\mathbf{X}^{(N)}, \mathbf{x})]^T$

$$\mathbf{k}_{\boldsymbol{\theta}}(\mathbf{X}, \mathbf{X}) = [k_{\boldsymbol{\theta}}(\mathbf{X}^{(i)}, \mathbf{X}^{(j)})]_{i,j=1}^{N}$$

$$\mathbf{y}(\mathbf{X}) = [f(\mathbf{X}^{(1)}), f(\mathbf{X}^{(2)}), \ldots, f(\mathbf{X}^{(N)})]^T$$

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

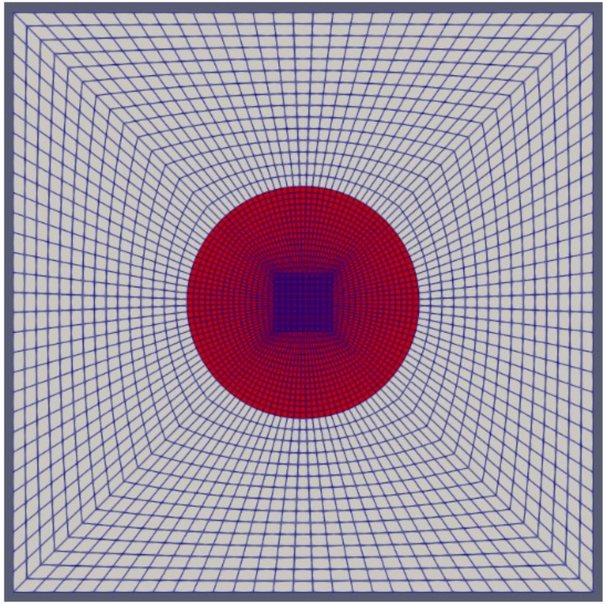
Fiber reinforced microstructure

- 4321 eight-node elements with 4 quadrature points
- Volume fraction of fiber is 12.56%
- Matrix material $C_1^{\mathrm{mat}} = D_1^{\mathrm{mat}} = 1$
- Fiber material $C_1^{\mathrm{fib}} = D_1^{\mathrm{fib}} \in [50, 150]$
- Deformations $\bar{\boldsymbol{U}} - \mathbf{I} \in [-0.3, 0.3]$
- 500 training snapshots (Sobol sequence)
- 1000 testing snapshots (uniformly random)
- P-PODGPR Error

$$\epsilon_{\bar{\mathbf{P}}} := \frac{\left\| \bar{\mathbf{P}}^{\mathrm{HF}} - \hat{\bar{\mathbf{P}}}^{\mathrm{RB}} \right\|_{\mathrm{F}}}{\left\| \bar{\mathbf{P}}^{\mathrm{HF}} \right\|_{\mathrm{F}}}$$

- Projection Error

$$\epsilon_{\bar{\mathbf{P}}} := \frac{\left\| \bar{\mathbf{P}}^{\mathrm{HF}} - \bar{\mathbf{P}}^{\mathrm{RB}} \right\|_{\mathrm{F}}}{\left\| \bar{\mathbf{P}}^{\mathrm{HF}} \right\|_{\mathrm{F}}}$$



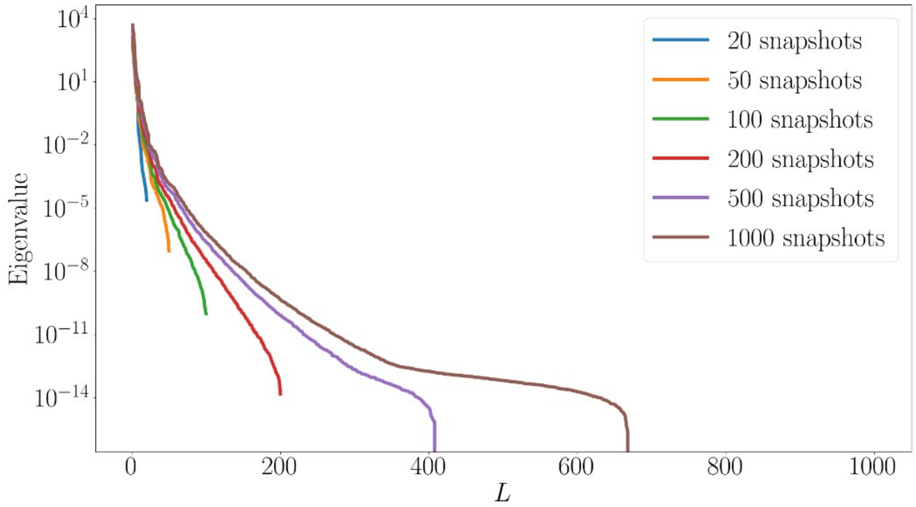
$$W(\boldsymbol{F}, \boldsymbol{\mu}) = C_1 (\mathrm{Tr}(\boldsymbol{F}^T \boldsymbol{F}) - 3 - 2 \ln (\det \boldsymbol{F})) + D_1 (\det \boldsymbol{F} - 1)^2$$

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Fiber reinforced microstructure

POD for different numbers of training snapshots



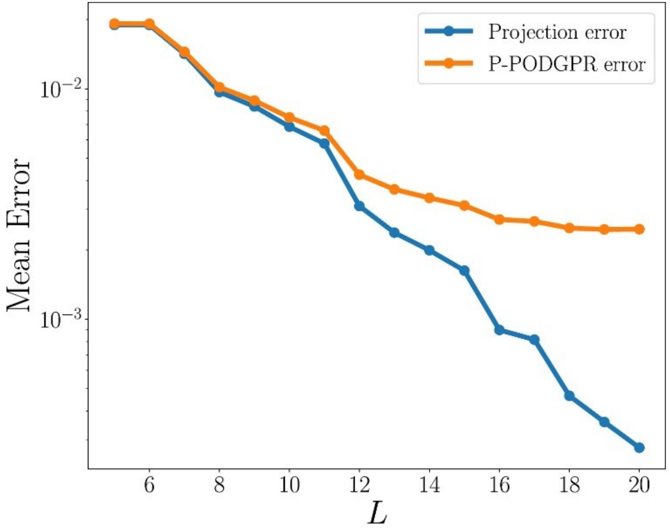
With $L = 20, \quad \dfrac{\sum_{i=1}^{L} \lambda_i}{\sum_{i=1}^{N} \lambda_i} = 0.9999$

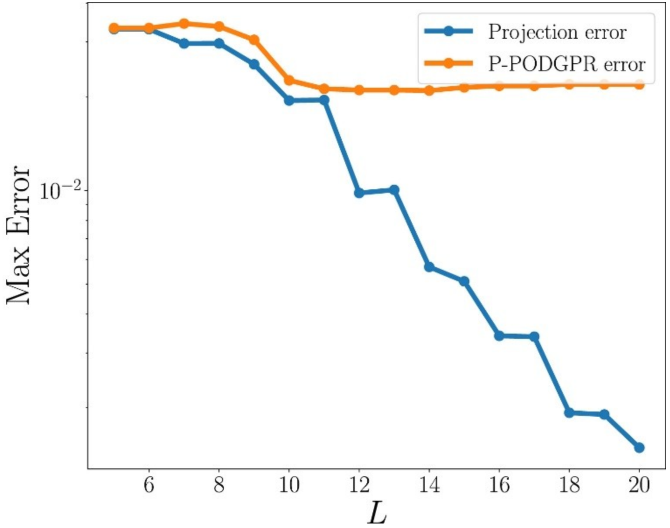
Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Fiber reinforced microstructure

Test error in effective stress for 200 training snapshots





Mean error ≈ 0.003

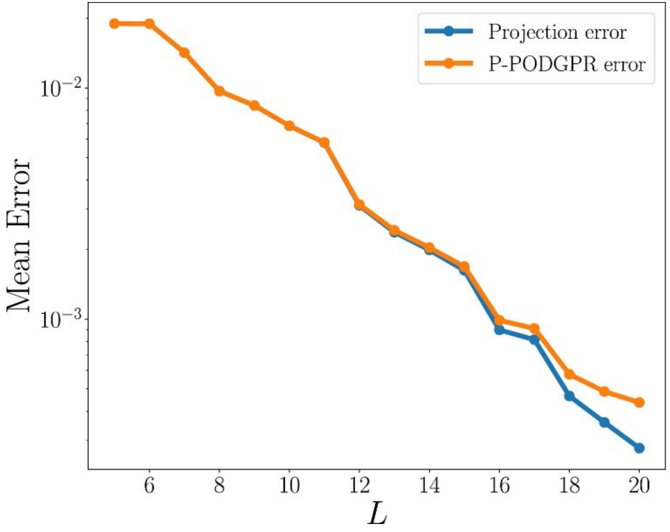
Max error ≈ 0.011

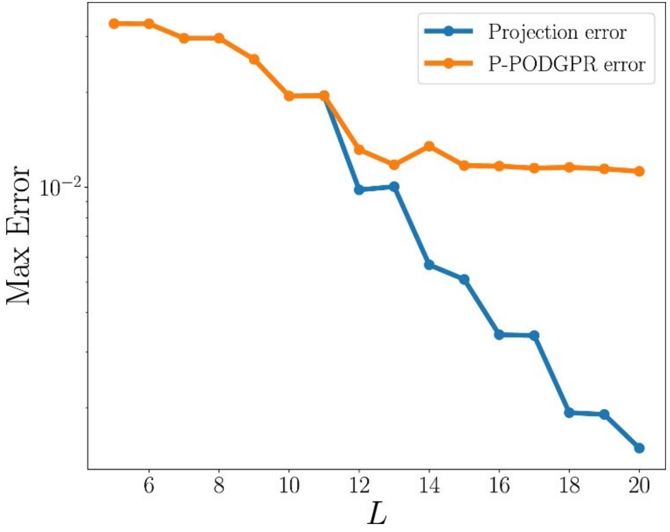
Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Fiber reinforced microstructure

Test error in effective stress for 500 training snapshots





Mean error ≈ 0.0004

Max error ≈ 0.01

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Fiber reinforced microstructure

- Comparison with a neural network (Multilayer Perceptron) that is trained with pairs of $\{\bar{\boldsymbol{U}}^{(i)}, \bar{\boldsymbol{P}}^{(i)}\}_{i=1}^{N}$ with $N = 500$ training snapshots
- Trained with Adam optimizer with learning rate of 10^{-4}, batch size of 32 and Mean Squared Error Loss function for different architectures
- ELU activation function was applied after every layer apart from last layer

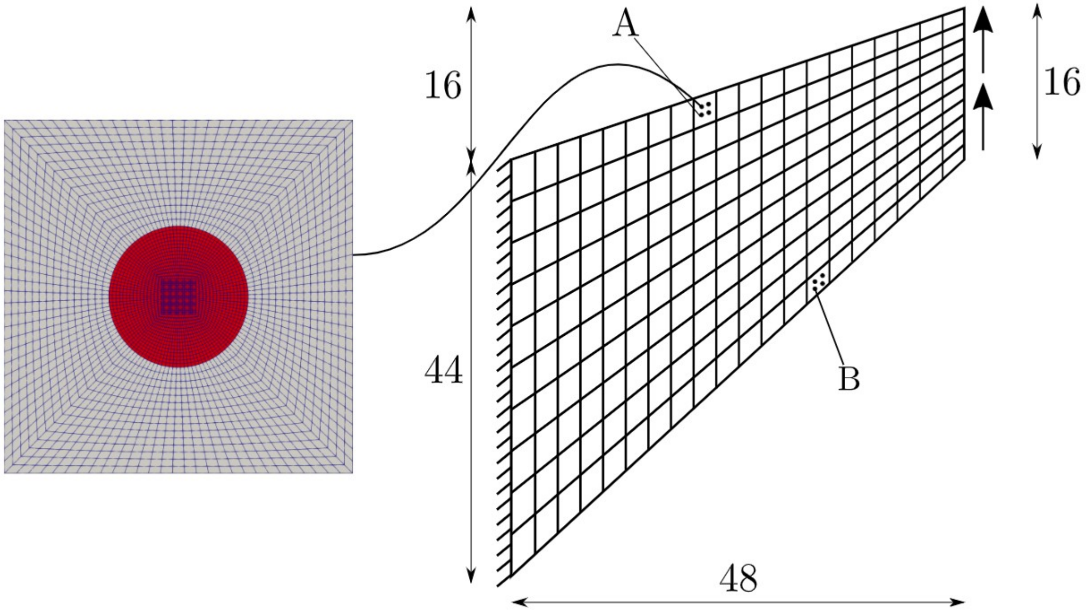
Architecture	Validation Loss	$\epsilon_{\bar{\mathbf{P}}}^{\text{mean}}$	$\epsilon_{\bar{\mathbf{P}}}^{\text{max}}$
$N_h = 1, N_n = 20$	1.7×10^{-6}	0.0077	0.0368
$N_h = 1, N_n = 50$	8.55×10^{-7}	0.0056	0.0289
$N_h = 2, N_n = 20$	5.36×10^{-7}	0.0047	**0.0176**
$N_h = 2, N_n = 50$	$\mathbf{2.97 \times 10^{-7}}$	**0.0039**	0.0206
$N_h = 2, N_n = 100$	7.91×10^{-7}	0.0052	0.0315

$\Rightarrow$ P-PODGPR outperforms all neural networks, while also being able to recover the microscopic stress field

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem



Numerical Examples

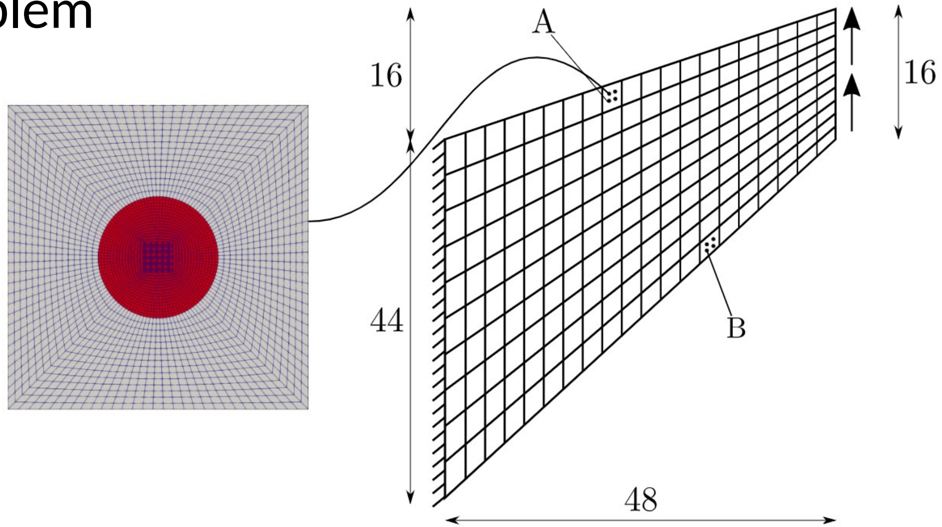
[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem

- 200 four-node elements with 4 quadrature points
- Vertical traction of 0.1 on right edge
- Left edge is fixed
- Matrix material $C_1^{\mathrm{mat}} = D_1^{\mathrm{mat}} = 1$
- Fiber material $C_1^{\mathrm{fib}} = D_1^{\mathrm{fib}} = 100$
- Surrogate model trained with 200 snapshots
- Relative error to compare microscopic stresses

$$\epsilon_{P_{yx}} := |P_{yx}^{\mathrm{FE2}} - P_{yx}^{\mathrm{ROM}}| / \langle |P_{yx}^{\mathrm{FE2}}| \rangle$$

$\langle |P_{yx}^{\mathrm{FE2}}| \rangle$: Averaged absolute stress



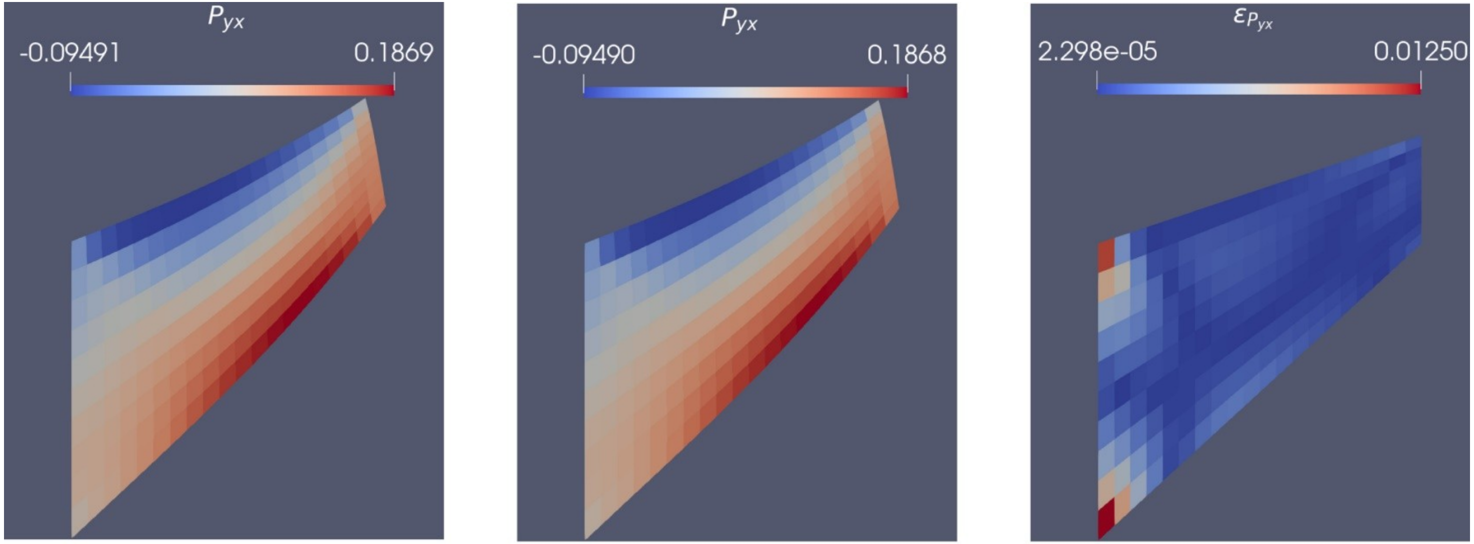
$$W(\boldsymbol{F}, \boldsymbol{\mu}) = C_1(\mathrm{Tr}(\boldsymbol{F}^T \boldsymbol{F}) - 3 - 2\ln(\det \boldsymbol{F})) + D_1(\det \boldsymbol{F} - 1)^2$$

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem

Macroscopic stress component P_{yx}



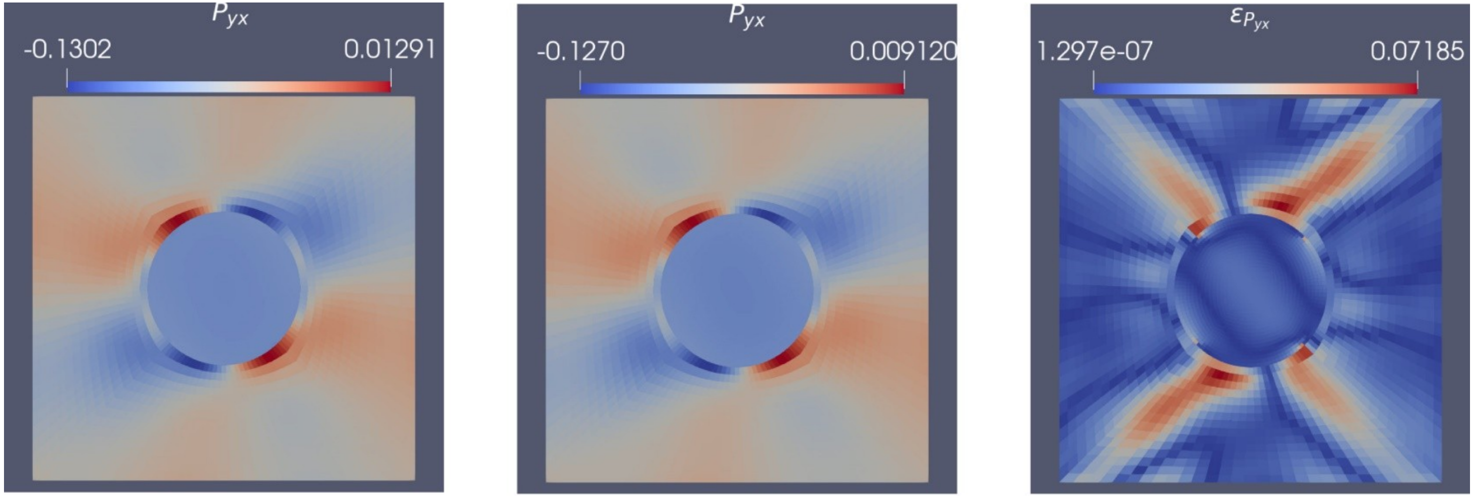
(a) FE^2 (b) FE with P-PODGPR (c) Relative error

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem

Microscopic stress component P_{yx} at point A



(a) FE^2 (b) FE with P-PODGPR (c) Relative error

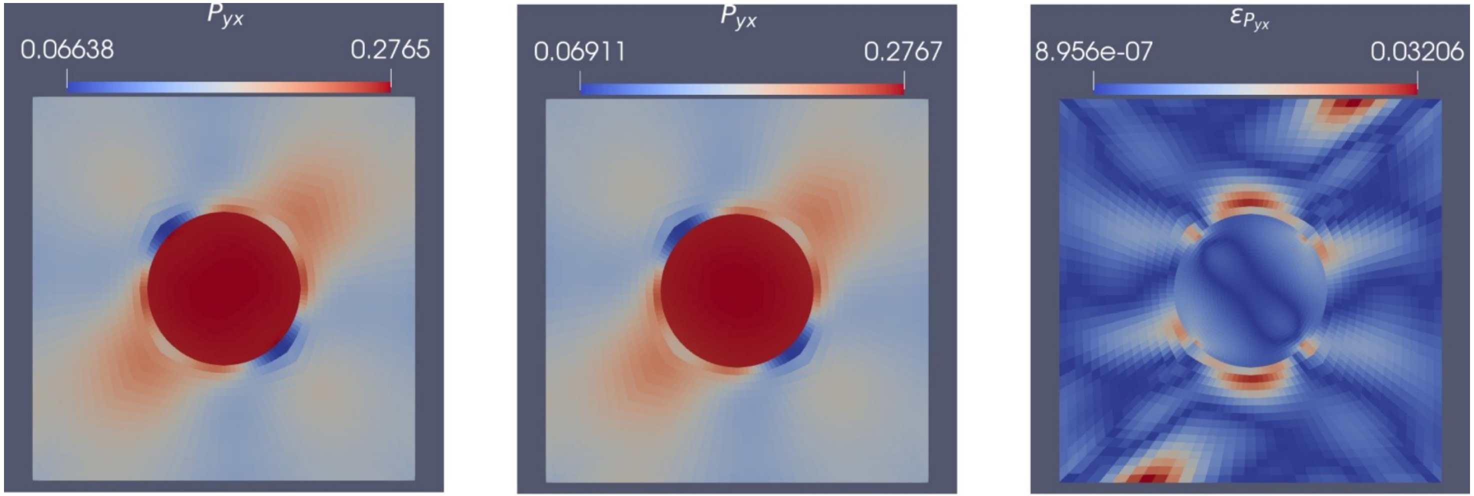
TU/e

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem

Microscopic stress component P_{yx} at point B



(a) FE^2 (b) FE with P-PODGPR (c) Relative error

Numerical Examples

[Guo, T., Rokoš, O., and Veroy, K., 2021, CMAME 384]

Two-scale Cook's membrane problem

	FE^2	FE with P-PODGPR
Snapshot generation	-	60 min
POD + GPR	-	1 min
Online simulation	4800 min	1 min

$\Rightarrow$ Online speedup of the order of 1000

Conclusion & Outlook

Summary

- Novel method based on non-intrusive reduced basis method
- Validated on 2D composite microstructure and multiscale application
- High accuracy and online speedup
- Sensitivities are available and can be used for optimization

Outlook

- Geometrical parameters
- Dissipative constitutive models (Plasticity, Damage)
- Test on 3D problems

Contact: Ondřej Rokoš (o.rokos@tue.nl)

References

[1] Guo, M., & Hesthaven, J. S. (2018). Reduced order modeling for nonlinear structural analysis using Gaussian process regression. *Computer Methods in Applied Mechanics and Engineering*, *341*, 807-826.
[2] Guo, T., Rokoš, O., & Veroy, K. (2021). Learning constitutive models from microstructural simulations via a non-intrusive reduced basis method. *Computer Methods in Applied Mechanics and Engineering*, *384*, 113924.
[3] Hernández, J. A., Oliver, J., Huespe, A. E., Caicedo, M. A., & Cante, J. C. (2014). High-performance model reduction techniques in computational multiscale homogenization. *Computer Methods in Applied Mechanics and Engineering*, *276*, 149-189.
[4] Kast, M., Guo, M., & Hesthaven, J. S. (2020). A non-intrusive multifidelity method for the reduced order modeling of nonlinear problems. *Computer Methods in Applied Mechanics and Engineering*, *364*, 112947.
[5] Radermacher, A., & Reese, S. (2016). POD-based model reduction with empirical interpolation applied to nonlinear elasticity. *International Journal for Numerical Methods in Engineering*, *107*(6), 477-495.
[6] Rasmussen, C. E. (2003, February). Gaussian processes in machine learning. In *Summer school on machine learning* (pp. 63-71). Springer, Berlin, Heidelberg.
[7] Soldner, D., Brands, B., Zabihyan, R., Steinmann, P., & Mergheim, J. (2017). A numerical study of different projection-based model reduction techniques applied to computational homogenisation. *Computational mechanics*, *60*(4), 613-625.
[8] Swischuk, R., Mainini, L., Peherstorfer, B., & Willcox, K. (2019). Projection-based model reduction: Formulations for physics-based machine learning. *Computers & Fluids*, *179*, 704-717.
[9] Yvonnet, J., & He, Q. C. (2007). The reduced model multiscale method (R3M) for the non-linear homogenization of hyperelastic media at finite strains. *Journal of Computational Physics*, *223*(1), 341-368.