PDE-based CNNs
with Morphological Convolutions

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# Basic Example

Modeling a heat process by a single layer CNN

<table>
<thead>
<tr>
<th>Heat equation</th>
<th>Analytic solution</th>
<th>Single layer CNN</th>
</tr>
</thead>
</table>
| $\begin{aligned}
\frac{\partial f}{\partial t} &= \nabla^2 f \\
f(0) &= f_0
\end{aligned}$ | $f(t) = G_t \ast f_0$ | $f_{\text{out}} = \sigma(K \ast f_0)$ |

with $G_t$ the heat kernel for time $t$  
With $\sigma$ a ReLU and $K$ a learnable kernel
Equivariance

Group CNNs

<table>
<thead>
<tr>
<th></th>
<th>Translation</th>
<th>Rotation</th>
<th>Scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spatial CNN</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SE(d) CNN</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>SIM(d) CNN</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>
Extending the domain
To a homogeneous space of the symmetry group

- Orientation Score Transform
- Can be learned
- More straightforward to design roto-translation equivariant CNNs
PDE-based CNN
An example segmentation network
PDE Layer

\[
\frac{\partial W_k}{\partial t} = -c_k \cdot \nabla W_k - |\Delta |^\alpha W_k \pm \|\nabla W_k\|^{2\alpha}, \quad W_k(\cdot, 0) = U_k(\cdot)
\]

<table>
<thead>
<tr>
<th>Goal</th>
<th>Transport</th>
<th>Regularization</th>
<th>Max pooling</th>
<th>Normalization</th>
<th>Combination</th>
</tr>
</thead>
<tbody>
<tr>
<td>PDE</td>
<td>Convection</td>
<td>Fractional diffusion</td>
<td>Dilation</td>
<td>Erosion</td>
<td>Create initial conditions for next set of PDEs</td>
</tr>
<tr>
<td>Numerical operation</td>
<td>Convolution</td>
<td>Morph. Convolution</td>
<td>ReLU</td>
<td>Linear combinations</td>
<td></td>
</tr>
</tbody>
</table>

Previous
Initial conditions

Solutions of PDEs after certain time

Initial conditions for next PDEs
Parameters

What will we be training?

\[ \frac{\partial W_k}{\partial t} = -c_k \cdot \nabla_{G_1} W_k - \left| \Delta_{G_2} \right|^\alpha W_k \pm \left\| \nabla_{G_3} W_k \right\|_{G_3}^{2\alpha} \]

- K convection vectors
- 3 metric tensor fields
  - Inducing metrics $d_{G_i}$ on the homogeneous space
- Design parameter: $\alpha \in \left( \frac{1}{2}, 1 \right]$
Constructing Solutions

Let $G$ be a Lie group (e.g. $SE(d)$)
Let $G/H$ be a homogeneous space (e.g. $SE(d)/(\{0\} \times SO(d−1))$)
$G$ acts on $G/H$ by $\odot$.
Let $f \in L^2, K_1 \in L^1(G/H, \mathbb{R})$.

• **Linear Convolution**

\[
(K_1 \ast f)(x) = \int_G K_1(h^{-1} \odot x) f(h \odot e) \, d\mu(h)
\]

• **Morphological Convolution**

\[
(K_2 \square f)(x) = \sup_{h \in G} K_2(h^{-1} \odot x) + f(h \odot e)
\]

• sup : dilation
• inf : erosion
Max Pooling

Morphological Convolution generalizes Max Pooling

For \( \alpha \downarrow \frac{1}{2} \) the solution kernel to the morphological part converges to

\[
K_t(x) = \begin{cases} 
0 & \text{if } d_{G_3}(e, x) \leq t, \\
-\infty & \text{else.}
\end{cases}
\]

\[
(K_t \Box f)(x) = \sup_{h \in G} K(h^{-1} \circ x) + f(h \circ e)
\]

\[
= \sup \{ f(h \circ e) \mid h \in G : d_{G_3}(e, h^{-1} \circ x) \leq t \}
\]

\[
= \sup \{ f(h \circ e) \mid h \in G : d_{G_3}(h \circ e, x) \leq t \}
\]

\[
= \sup_{y \in B(x, t)} f(y)
\]

For \( \alpha > \frac{1}{2} \) : “soft” max pooling
Geometric Interpretation

\[
\frac{\partial W_k}{\partial t} = -c_k \cdot \nabla G_1 W_k - \Delta G_2^\alpha W_k \pm \left\| \nabla G_3 W_k \right\|_{G_3}^{2\alpha}
\]
Maintaining Equivariance

Conditions on the kernels?

- Linear convolution

\[ \forall g \in G: \quad (K * f)(g \odot x) = \left( K * (y \mapsto f(g \odot y)) \right)(x) \]

\[ \iff \]

\[ \forall g, h \in G \quad \forall x \in G/H: \quad K(hg \odot x) = K(gh \odot x) \]

- Same for morphological convolution

- Kernel symmetries are required!
First Experiment

Adding morphological convolution to a retinal segmentation network

• Spatial CNN
  • 6 convolution layers
  • 34580 parameters

• SE(2) group CNN
  • Lift layer, 4 convolution layers, projection layer
  • 33916 parameters

• SE(2) group CNN with single morph. convolution
  • Lift layer, 4 convolution layers, dilation layer, projection layer
  • 33916 parameters
  • Fixed morph. convolution kernel for $\alpha = \frac{2}{3}$
First Experiment

Performance improvement
Current and Future Work

- TensorFlow implementation
  - experiments
- Layer architectures
- Geometric interpretability
- Probabilistic interpretability
- Integrate PDE framework for geometric equivariant processing of orientation scores (2005-now)
Concluding remarks

Geometric PDE framework for CNNs

• Improved performance over state-of-the-art G-CNNs
  • by inclusion of single PDE-based morphological convolution layer
• Problem symmetry integral part of the design

Deep Learning

Mathematics

• Also see 2 talks in IHP Paris search: “Remco Duits IHP”
• Chapter 2 of my MSc thesis https://bmnsmets.com/publication/smets2019msc/