Randomized Model Order Reduction

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joint work with
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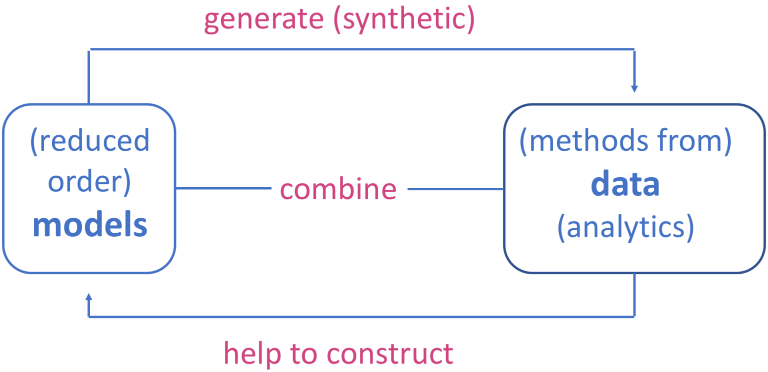
4TU-AMI Symposium "Reducing dimensions in Big Data: Model Order Reduction in action"

Data and Model Order Reduction



Motivation/Outline

- Motivation: Randomized methods have got a steadily growing deal of attention in recent years, especially for problems in large-scale data analysis.
  Two most important benefits:
  - They can result in faster algorithms, either in worst-case asymptotic theory and/or numerical implementation,
  - they allow very often for (novel) tight error estimators
- Topic of this talk: Show how we can benefit from randomized methods in model order reduction

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- Topic of this talk: Show how we can benefit from randomized methods in model order reduction
- Outline:
  1. Introduction to projection-based model order reduction
  2. Construct reduced spaces via randomized methods
  3. Randomized a posteriori error estimator for projection-based model reduction

Parametrized Partial Differential Equation

- Parameter vector $\mu \in \mathcal{P}$; compact parameter set $\mathcal{P} \subset \mathbb{R}^P$
- Parametrized PDE: Given any $\mu \in \mathcal{P}$, find $u(\mu) \in X$, s.th.

$$A(\mu)u(\mu) = f(\mu) \quad \text{in } X'.$$

- $\Omega \subset \mathbb{R}^3$: bounded domain with Lipschitz boundary $\partial\Omega$
- $H_0^1(\Omega)^d \subset X \subset H^1(\Omega)^d$ $(d = 1, 2, 3)$; X': dual space
- $A(\mu) : X \to X'$: inf-sup stable, continuous linear differential operator
- $f(\mu) : X \to \mathbb{R}$: continuous linear form

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- High-dimensional discretization:
- Introduce high-dimensional FE space $X^{\mathcal{N}} \subset X$ with $\dim(X^{\mathcal{N}}) = \mathcal{N}$ (assume small discretization error)
- High-dimensional approximation: Given any $\mu \in \mathcal{P}$, find $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$, s.th.

$$A(\mu)u^{\mathcal{N}}(\mu) = f(\mu) \quad \text{in } X^{\mathcal{N}'}.$$

- Issue: Require $u^{\mathcal{N}}(\mu)$ in real time and/or for many $\mu \in \mathcal{P}$.

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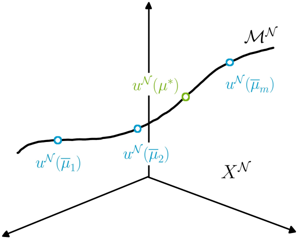
$$\underline{A}(\mu)\underline{u}^{\mathcal{N}}(\mu) = \underline{f}(\mu) \quad \underline{A}(\mu) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}, \underline{f}(\mu) \in \mathbb{R}^{\mathcal{N}}.$$

- Issue: Require $u^{\mathcal{N}}(\mu)$ in real time and/or for many $\mu \in \mathcal{P}$.

Projection-based model order reduction: key concept

- Exploit: $u^{\mathcal{N}}(\mu)$ belongs to "solution manifold" $\mathcal{M}^{\mathcal{N}} = \{u^{\mathcal{N}}(\mu) \,|\, \mu \in \mathcal{P}\} \subset X^{\mathcal{N}}$ of typically very low dimension
- Offline: Construct reduced space $X^N \subset X^{\mathcal{N}}$ from solutions $u^{\mathcal{N}}(\bar{\mu}_i)$, $i = 1, \dots, N$ (e.g. by a Greedy algorithm, Proper Orthogonal Decomposition,...)



- Online: Galerkin projection on X^N: Given any $\mu^* \in \mathcal{P}$, find $u^N(\mu^*) \in X^N$, s.th.

$$A(\mu^*)u^N(\mu^*) = f(\mu^*) \quad \text{in } X^{N\prime}.$$

- If $\mathcal{M}^{\mathcal{N}}$ is smooth, $N \ll \mathcal{N}$ already yields a very accurate approximation. ([DeVore Petrova Wojtaszczyk 13])

For an overview on model order reduction see for instance [Benner, Cohen, Ohlberger, Willcox 2017].

Proper Orthogonal Decomposition via SVD

- Introduce finite dimensional training set $\mathcal{P}^{train} \subset \mathcal{P}$ of dimension n_{train}
- Compute solutions $u^{\mathcal{N}}(\mu)$ for all $\mu \in \mathcal{P}^{train}$
- Store coefficients in a so-called snapshot matrix:
  $Y = [\underline{u}^{\mathcal{N}}(\mu_1)| \dots |\underline{u}^{\mathcal{N}}(\mu_{n_{train}})]$
- Perform Singular Value decomposition: $Y = U\Sigma V^*$
- Use first N left singular vectors to define reduced space

Algorithm 1: Greedy algorithm

input : finite dimensional training set $\mathcal{P}^{train} \subset \mathcal{P}$, tolerance *tol*
output: S_N, X^N

Initialize: $S_1 = \emptyset$, $X^0 = \{0\}$, $\Delta_0(\mu) = \|u^{\mathcal{N}}(\mu)\|_X$
for $N = 1 : N_{max}$ **do**

$$\text{Find: } \mu_N = \arg \max_{\mu \in \mathcal{P}^{train}} \Delta_{N-1}(\mu). \quad (\|u^{\mathcal{N}}(\mu) - u^N(\mu)\|_X \leqslant \Delta_N(\mu))$$

Solve for $u^{\mathcal{N}}(\mu_N)$.

$$\text{Extend: } S_N = S_{N-1} \cup \mu_N \text{ and } X^N = \text{span}\{u^{\mathcal{N}}(\mu_1), \ldots, u^{\mathcal{N}}(\mu_N)\}.$$

Compute $\Delta_N(\mu)$ for all $\mu \in \mathcal{P}^{train}$.

if $\arg \max_{\mu \in \mathcal{P}^{train}} \Delta_N(\mu) \leqslant tol$ **then**

break

end

end

Certification via a posteriori error bound

- A posteriori error estimator is important both
  - to construct reduced order models via the greedy
  - certify the approximation in the online stage: how large is the error (in some quantity of interest)?

Proposition (A posteriori error bound)

The error estimator $\Delta_N(\mu) = \beta_{LB}(\mu)^{-1}\|f(\mu) - A(\mu)u^N(\mu)\|_{X^{\mathcal{N}\prime}}$ *with* $\beta_{LB}(\mu) \leqslant \beta_{\mathcal{N}}(\mu)$ *satisfies*

$$\|u^{\mathcal{N}}(\mu) - u^N(\mu)\|_X \leqslant \Delta_N(\mu) \leqslant \frac{\gamma_{\mathcal{N}}(\mu)}{\beta_{LB}(\mu)}\|u^{\mathcal{N}}(\mu) - u^N(\mu)\|_X,$$

where $\beta_{\mathcal{N}}(\mu) := \inf_{v\in X^{\mathcal{N}}} \sup_{w\in X^{\mathcal{N}}} \frac{\langle A(\mu)v, w\rangle}{\|v\|_X\|w\|_X}$ *and* $\gamma_{\mathcal{N}}(\mu) = \sup_{v\in X^{\mathcal{N}}} \sup_{w\in X^{\mathcal{N}}} \frac{\langle A(\mu)v, w\rangle}{\|v\|_X\|w\|_X}$.

- Problem: Good estimate of inf-sup often computationally infeasible
  $\longrightarrow$ constant-free randomized error estimators

Constructing reduced order models via methods from randomized numerical linear algebra

For an overview on algorithms in randomized numerical linear algebra see for instance: [Halko et al 2011], [Mahoney 2011], [Drineas, Mahoney, 2016]

Randomized SVD

- Goal: Given a snapshot matrix $Y \in \mathbb{R}^{\mathcal{N} \times n_{train}}$ and an integer k find an orthonormal matrix Q of rank k such that $Y \approx QQ^*Y$.
- Approach:
- Draw k random vectors $r_j \in \mathbb{R}^{n_{train}}$ (say standard Gaussian)
- Form sample vectors $s_j = Yr_j \in \mathbb{R}^{\mathcal{N}} \quad j = 1, \ldots, k$.
- Orthonormalize $s_j \longrightarrow q_j, = 1, \ldots, k$ and define $Q = [q_1, \ldots, q_k]$
- Result: If Y has exactly rank k then $q_j, = 1, \ldots, k$ span the range of Y at probability 1. But also in the general case $q_j, = 1, \ldots, k$ often perform nearly as good as the k leading left singular vectors of Y
- Compute randomized SVD:
- Form $C = Q^*Y$ which yields $Y \approx QC$
- Compute SVD of of the small matrix $C = \widetilde{U}\Sigma V^*$ and set $U = Q\widetilde{U}$

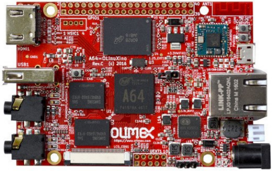
References in MOR: Hochman et al 2014, Alla, Kutz 2015, Balabanov, Nouy 2018

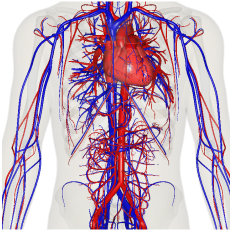
Reduced order modelling for large-scale problems

Limitations of standard model order reduction approach:

- Curse of parameter dimensionality: many parameters require prohibitively large reduced spaces
- No topological flexibility (although geometric variation is possible)
- Possibly high computational costs in the offline stage

$\longrightarrow$ Localized model order reduction





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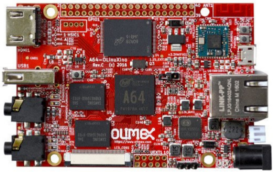
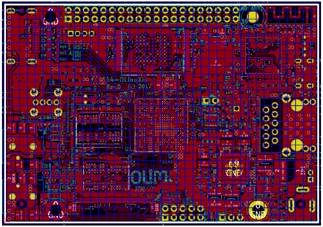
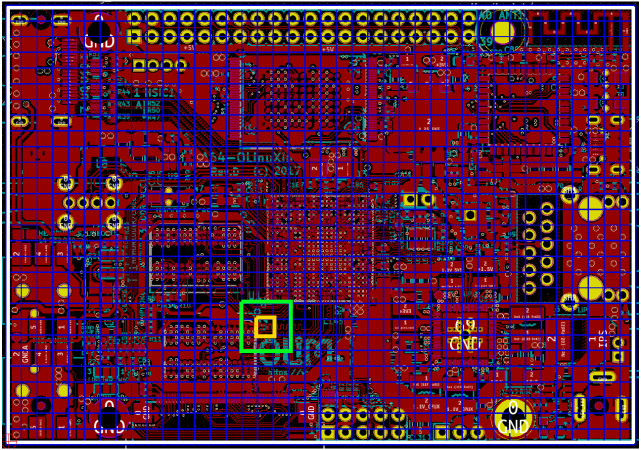




Domain decomposition and oversampling

For each small subdomain ω_i (e.g. ω_{816}, yellow)
we introduce an oversampling domain ω_i^* (e.g. ω_{816}^*, green)



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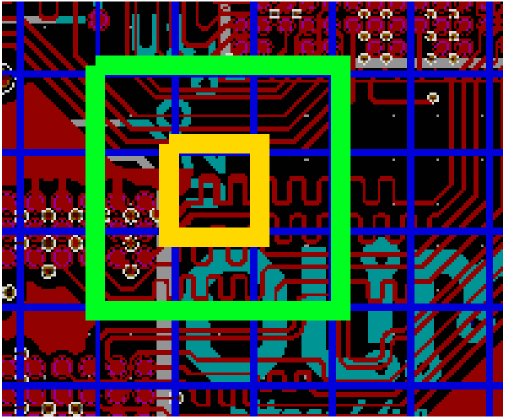

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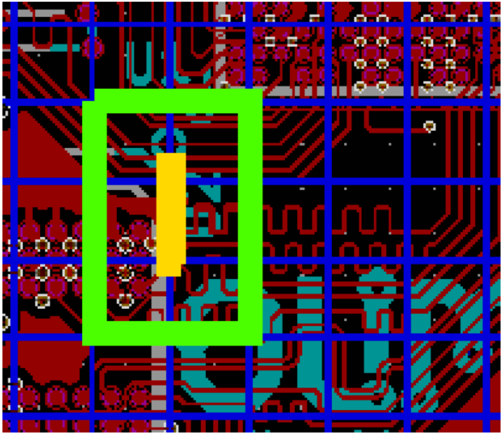
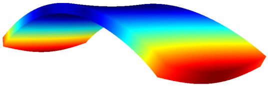


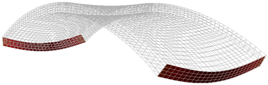
Constructing local reduced models via a transfer operator

Introduce transfer operators T_i:

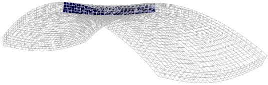
- ... acts on the space of local solutions of the PDE and maps values ζ on $\partial\omega_i^*$ to ω_i



- ... by solving the PDE locally with Dirichlet boundary values ζ



- ... and restricting the local solution to ω_i



Constructing optimal local spaces via transfer operators

- Key observation: Global solution u satisfies $u|_{\omega_i} = T_i(u|_{\partial\omega_i^*})$

$\Longrightarrow$ Construct local reduced spaces that approximate $\mathrm{range}(T_i)$

- ϕ_j: the left singular vectors of T_i; σ_j: singular values of T_i
- The reduced space

$$R_i^{opt,n} := \text{span}\{\phi_1, \ldots, \phi_n\}$$

is the optimal space and minimizes the approximation error among all spaces of the same dimension

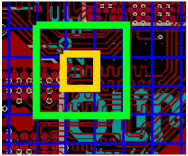
- The error satisfies:

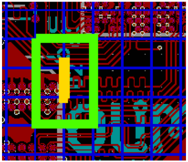
$$\| T_i - P_{R_i^{opt,n}} T_i \| = \sigma_{n+1}, \quad P_{R_i^{opt,n}} : \text{ orthogonal projection on } R_i^{opt,n}$$

References: Babuska, Lipton, MMS, 2011; Smetana, Patera, SISC, 2016

Approximating optimal local spaces with Randomized Linear Algebra

- Prescribe random boundary conditions; in detail choose every coeffcient of a FEM basis function on $\partial\omega_i^*$ as a (mutually independent) Gaussion random variable with zero mean and variance one
- Solve PDE for random boundary conditions numerically and store evaluation of local solution of PDE $u|_{\omega_i}$.
- Define reduced space $R^n_{i,rand}$ as the span of n such evaluations $u|_{\omega_i}$.





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Questions: What is the quality of such an approximation?
(How) can we determine the dimension of the reduced space for a given tolerance?

References: Buhr, Smetana, SISC, 2018

Probalistic a priori error bound[1]

Proposition (A priori error bound (Buhr, Smetana 18))

$T : S \to R$ transfer operator as above, p oversampling parameter, $n, p \geqslant 2$

$$\mathbb{E}\| T_i - P_{R_{i,rand}^{n+p}} T_i \| \leqslant \underbrace{\sqrt{\frac{\lambda_{max}^{M_R}}{\lambda_{min}^{M_R}} \frac{\lambda_{max}^{M_S}}{\lambda_{min}^{M_S}}} \left\{ \left(1 + \frac{\sqrt{n}}{\sqrt{p-1}}\right) \sigma_{n+1} + \frac{e\sqrt{n+p}}{p} \left(\sum_{j>n} \sigma_j^2 \right)^{1/2} \right\}}_{\sim c\sqrt{n}\sigma_{n+1}}$$

Optimal convergence rate achieved via SVD:

$$\| T_i - P_{R_i^{opt,n}} T_i \| = \sigma_{n+1}$$

[1] based on results in [Halko, Martinsson, Tropp 11]

Probablistic a posteriori error bound[2]

Proposition (Probablistic a posteriori error bound (Buhr, Smetana 2018))

$\{\underline{r}^{(j)} \,:\, j = 1, 2, ..., n_t\}$: *standard Gaussian vectors*

Define $$\Delta(n_t, \delta_{\mathrm{tf}}) := \frac{c_{\mathrm{est}}(n_t, \delta_{\mathrm{tf}})}{\sqrt{\lambda_{min}^{M_S}}} \max_{j \in 1,\dots,n_t} \left(\| T_i \underline{r}^{(j)} - P_{R^n_{i,rand}} T_i \underline{r}^{(j)} \| \right)$$

Then there holds

$$\| T_i - P_{R^n_{i,rand}} T_i \| \leqslant \Delta(n_t, \delta_{\mathrm{tf}}) \leqslant \left(\frac{\lambda_{max}^{M_S}}{\lambda_{min}^{M_S}} \right)^{1/2} c_{\mathrm{eff}}(n_t, \delta_{\mathrm{tf}}) \| T_i - P_{R^n_{i,rand}} T_i \|$$

with a probability of at least $1 - \delta_{\mathrm{tf}}$.

[2] Estimator extends results in [Halko, Martinsson, Tropp 11]; effectivity bound new

Adaptive randomized range finder

- Input: Select tolerance tol, failure probability $\delta_{algofail}$
- While $\Delta(n_t, \delta_{tf}) > tol$
  - Generate random boundary values on $\partial\omega_i^*$
  - Apply transfer operator T_i to random boundary conditions
  - Add new solution to $R^n_{i,rand}$
  - Orthonormalize solutions
  - Update a posteriori error estimator
- Output: $R^n_{i,rand}$ such that $\| T_i - P_{R^n_{i,rand}} T_i \| \leqslant tol$ with probability at least $1 - \delta_{algofail}$

Numerical Experiments for analytic test problem

Numerical Experiments: interfaces

- local (oversampling) domain $\omega^* := (-1,1) \times (0,1)$
- Consider PDE: $-\Delta u = 0$ in ω^*
- Goal: Construct reduced space on interface Γ_{in}

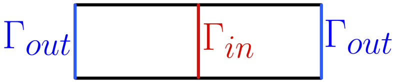
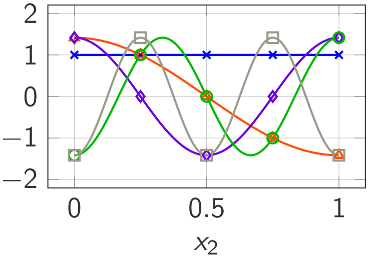


Figure: ω^*

Heat conduction: $-\Delta u = 0$ on $\omega^* = (-1, 1) \times (0, 1)$



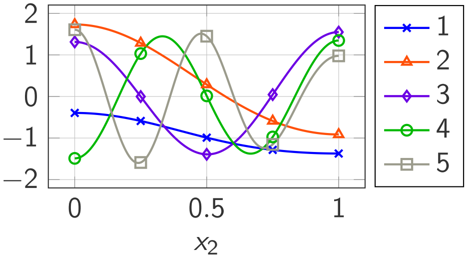
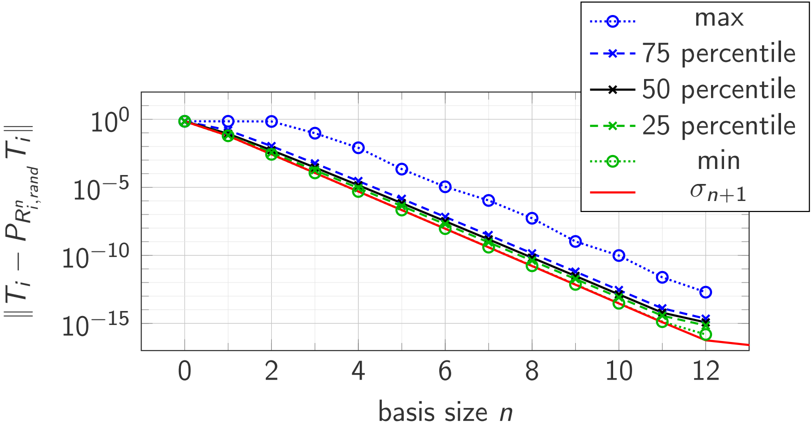


Figure: optimal basis — basis generated by randomized range finder algorithm

Heat conduction: $-\Delta u = 0$ on $\omega^* = (-1, 1) \times (0, 1)$



Heat conduction: $-\Delta u = 0$ on $\omega^* = (-1, 1) \times (0, 8)$

CPU times

Properties of basis generation

	randomized	Scipy/ARPACK
(resulting) basis size n	39	39
operator evaluations	59	79
adjoint operator evaluations	0	79
execution time in s (without factorization)	20.4 s	47.9 s

Table: CPU times; Target accuracy $\texttt{tol} = 10^{-4}$, number of testvectors $n_t = 20$, failure probability $\delta_{\mathrm{algofail}} = 10^{-15}$; unknowns of corresponding problem 638,799

Numerical Experiments for a transfer operator with slowly decaying singular values

Numerical Experiments: subdomains

- local (oversampling) domain $\omega^* := (-2,2) \times (-0.25, 0.25) \times (-2,2)$
- Consider PDE: linear elasticity in ω^* (isotropic, homogeneous)
- Goal: Construct reduced space on $\omega = (-0.5, 0.5) \times (-0.25, 0.25) \times (-0.5, 0.5)$

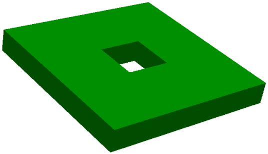


Figure: $\omega^* \backslash \omega$

Linear elasticity on $\Omega := (-2, 2) \times (-0.5, 0.5) \times (-2, 2)$

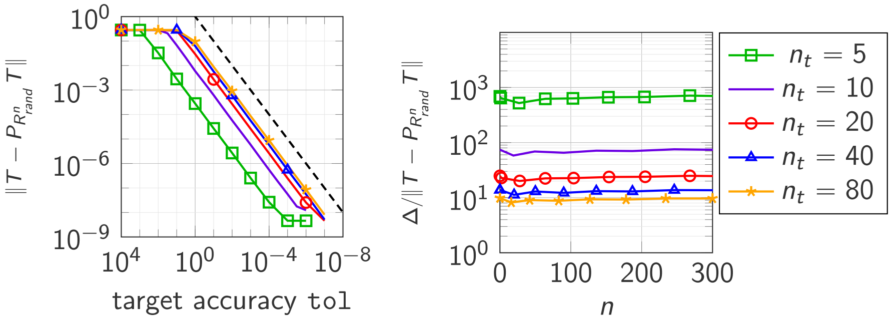


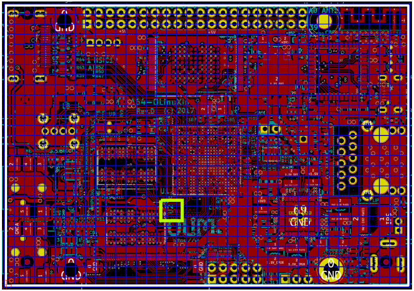
Figure: Convergence behavior of adaptive algorithm (left) and effectivity of a posteriori error estimator $\Delta / \|T - P_{R^n_{rand}} T\|$ (right) for increasing number of test vectors n_t.

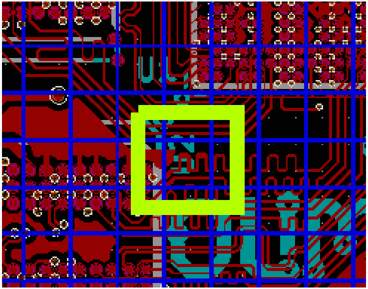
Olimex A64

- 1.2 GHz quad-core ARM CPU
- 1 GB of RAM
- open hardware
- designed with KiCAD

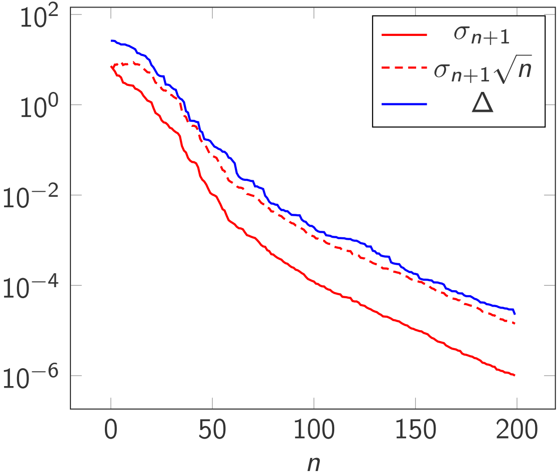
Results by Andreas Buhr

Domain 816

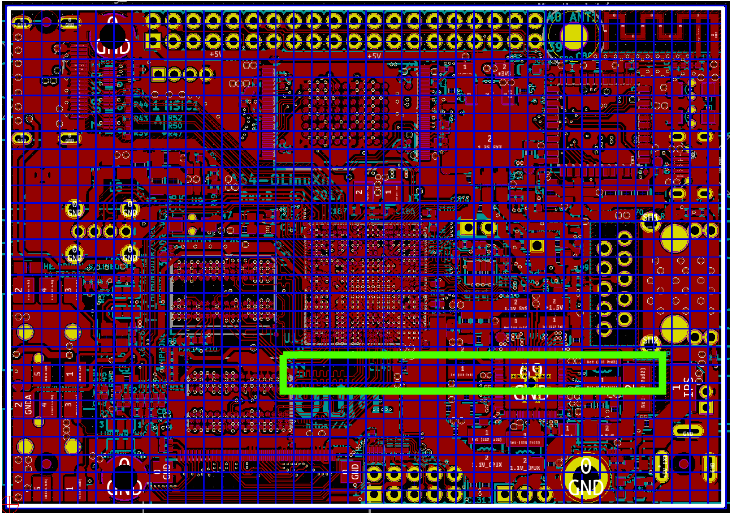




Decay of Singular Values for T_{816}



Error Estimator Decay



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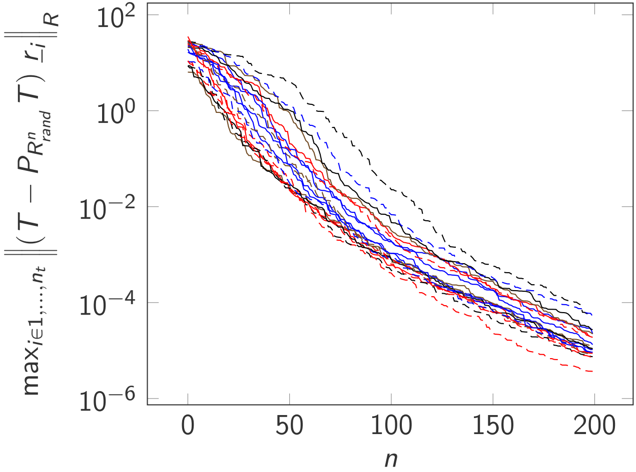

Randomized residual-based error estimators for parametrized equations

(joint work with A. T. Patera and O. Zahm)

Randomization within error estimation:

- Cao, Petzold 2004, Homescu, Petzold, Serban 2005
- Drohmann, Carlberg 2015, Trehan, Carlberg, and Durlofsky 2017
- Manzoni, Pagani, Lassila 2016
- Janon, Nodet, Prieur 2016
- Zahm, Nouy 2016
- Giraldi, Nouy 2017
- Balabanov, Nouy 2018

- Goal: Develop a posteriori error estimator for projection-based model order reduction that does not contain constants whose estimation is expensive (inf-sup constant)

- Setting: We query a finite number of parameters in the online stage for which we want to estimate the approximation error.

- Approach: Exploit concentration inequalities:

Proposition (Concentration inequality, Johnson-Lindenstrauss)

Choose rows of matrix $\Phi \in \mathbb{R}^{K \times \mathcal{N}}$ say as K independent copies of standard Gaussian random vectors scaled by $1/\sqrt{K}$ and let $\mathcal{S} \subset \mathbb{R}^{\mathcal{N}}$ be a finite set. Moreover, assume $K \geqslant (C(z)/\varepsilon^2)\log(\#\mathcal{S}/\delta)$. Then we have

$$\mathbb{P}\left\{(1-\varepsilon)\|x-y\|_2^2 \leqslant \|\Phi x - \Phi y\|_2^2 \leqslant (1+\varepsilon)\|x-y\|_2^2 \quad \forall x, y \in \mathcal{S}\right\} \geqslant 1-\delta.$$

see for instance [Boucheron, Lugosi, Massart 2012], [Vershynin 2012], [Vershynin 2018]

Assumptions on random vector

- $Z \in \mathbb{R}^{\mathcal{N}}$: random vector such that

$$\|v\|_{\Sigma}^2 = v^T \Sigma v = \mathbb{E}((Z^T v)^2) \quad \forall v \in \mathbb{R}^{\mathcal{N}},$$

where Σ is matrix e.g. associated with H^1- or L^2-inner product or a quantity of interest

$\Longrightarrow$ $(Z^T v)^2$ is an unbiased estimator of $\|v\|_{\Sigma}^2$

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- For simplicity: Assume $Z \sim \mathcal{N}(0, \Sigma)$ is a Gaussian vector with zero mean and covariance matrix Σ

Assumptions on random vector

- $Z \in \mathbb{R}^{\mathcal{N}}$: random vector such that

$$\|v\|_{\Sigma}^2 = v^T \Sigma v = \mathbb{E}((Z^T v)^2) \quad \forall v \in \mathbb{R}^{\mathcal{N}},$$

  where Σ is matrix e.g. associated with H^1- or L^2-inner product or a quantity of interest

$\Longrightarrow$ $(Z^T v)^2$ is an unbiased estimator of $\|v\|_{\Sigma}^2$

- For simplicity: Assume $Z \sim \mathcal{N}(0, \Sigma)$ is a Gaussian vector with zero mean and covariance matrix Σ
- $Z_1, \dots, Z_K$: K independent copies of Z
- Consider the following (unbiased) Monte-Carlo estimator of $\|v\|_{\Sigma}^2$

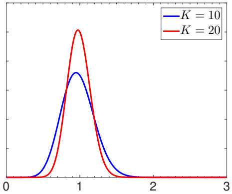
$$\frac{1}{K} \sum_{i=1}^{K} (Z_i^T v)^2.$$

Proposition (Concentration inequality for set of vectors)

Given a finite set of parameters $\mathcal{S} = \{\mu_1, \ldots, \mu_S\} \subset \mathcal{P}$, *a failure probability* $0 < \delta < 1$, $w \in \mathbb{R}$, $w > \sqrt{e}$, *we have for* $\underline{e}(\mu_j) = \underline{u}^{\mathcal{N}}(\mu_j) - \underline{u}^N(\mu_j)$,

$$K \geqslant \frac{\log(\#\mathcal{S}) + \log(\delta^{-1})}{\log(w/\sqrt{e})} \qquad \textit{that}$$

$$\mathbb{P}\left\{ \frac{\|\underline{e}(\mu_j)\|_{\Sigma}^2}{w^2} \leqslant \frac{1}{K}\sum_{i=1}^{K}(Z_i^T \underline{e}(\mu_j))^2 \leqslant w^2 \|\underline{e}(\mu_j)\|_{\Sigma}^2, \ \forall \mu_j \in \mathcal{S} \right\} \geqslant 1 - \delta.$$



- chi-squared distribution
- concentration around 1 (that means error estimator has perfect effectivity 1)

Proposition (Concentration inequality for set of vectors)

Given a finite set of parameters $\mathcal{S} = \{\mu_1, \dots, \mu_S\} \subset \mathcal{P}$, a failure probability $0 < \delta < 1$, $w \in \mathbb{R}$, $w > \sqrt{e}$, we have for $\underline{e}(\mu_j) = \underline{u}^{\mathcal{N}}(\mu_j) - \underline{u}^N(\mu_j)$,

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	$w = 2$	$w = 3$	$w = 4$	$w = 5$	$w = 10$
$\#\mathcal{S} = 1$	24	8	6	5	3
$\#\mathcal{S} = 100$	48	16	11	9	6
$\#\mathcal{S} = 1000$	60	20	13	11	7
$\#\mathcal{S} = 10^6$	96	31	21	17	11

Table: Values for K that guarantee (1) for all $\mu_j \in \mathcal{S}$ with $\delta = 10^{-2}$.

Proposition (Concentration inequality for set of vectors)

Given a finite set of parameters $\mathcal{S} = \{\mu_1, \ldots, \mu_S\} \subset \mathcal{P}$, *a failure probability* $0 < \delta < 1$, $w \in \mathbb{R}$, $w > \sqrt{e}$, *we have for* $\underline{e}(\mu_j) = \underline{u}^{\mathcal{N}}(\mu_j) - \underline{u}^{N}(\mu_j)$,

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Define $\Delta(\mu) := \left(\frac{1}{K}\sum_{i=1}^{K}(Z_i^T\underline{e}(\mu))^2\right)^{1/2}$

Problem: estimator $\Delta(\mu) = \left(\frac{1}{K}\sum_{i=1}^{K}(Z_i^T(\underline{u}^{\mathcal{N}}(\mu_j) - \underline{u}^{N}(\mu_j)))^2\right)^{1/2}$

involves high-dimensional finite element solution

$\Longrightarrow$ Computationally infeasible in the online stage

A fast-to-evaluate randomized error estimator

- Exploit error residual relationship

$$Z_i^T \underline{e}(\mu) = Z_i^T \underline{A}(\mu)^{-1} \underbrace{(\underline{f}(\mu) - \underline{A}(\mu)\underline{u}^N(\mu))}_{\text{residual } \underline{r}(\mu):=} = (\underbrace{\underline{A}(\mu)^{-T} Z_i}_{\text{dual problem}})^T \underline{r}(\mu)$$

- Define solutions of dual problems with random right-hand sides Z_i:

$$\underline{y}_i^{\mathcal{N}}(\mu) := \underline{A}(\mu)^{-T} Z_i$$

- Approximation of the dual solutions via model order reduction:

$$\underline{y}_i^{\mathcal{N}}(\mu) \approx \underline{\widetilde{y}}_i(\mu) \in \widetilde{\mathcal{Y}} \subset X^{\mathcal{N}},$$

where $\widetilde{\mathcal{Y}}$ dual reduced space
- Define fast-to-evaluate randomized error estimator

$$\widetilde{\Delta}(\mu) := \left(\frac{1}{K} \sum_{i=1}^{K} (\underline{\widetilde{y}}_i(\mu)^T \underline{r}(\mu))^2 \right)^{1/2}$$

A fast-to-evaluate randomized error estimator

Proposition

Choose $S \in \mathbb{N}$ in the offline stage. Then, in the online stage for any given $w > \sqrt{e}$ and $\delta > 0$ we have for S different parameters values μ_j, $j = 1, \ldots, S$ in a finite parameter set $\mathcal{S} = \{\mu_1, \ldots, \mu_S\}$ and

$$K \geqslant \frac{\log(S) + \log(\delta^{-1})}{\log(w/\sqrt{e})} \qquad \textit{that} \qquad \tilde{\Delta}(\mu_j) := \left(\frac{1}{K} \sum_{i=1}^{K} (\underline{\tilde{y}}_i(\mu_j)^T \underline{r}(\mu_j))^2 \right)^{1/2}$$

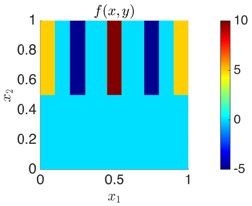
satisfies

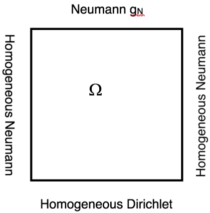
$$\mathbb{P}\left\{ (\alpha w)^{-1} \tilde{\Delta}(\mu_j) \leqslant \|\underline{e}(\mu_j)\|_{\Sigma} \leqslant (\alpha w)\, \tilde{\Delta}(\mu_j), \quad \mu_j \in \mathcal{S}, \right\} \geqslant 1 - \delta,$$

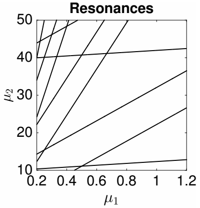
where

$$\alpha = \max_{\mu \in \mathcal{P}} \left(\max \left\{ \frac{\Delta(\mu)}{\tilde{\Delta}(\mu)}, \frac{\tilde{\Delta}(\mu)}{\Delta(\mu)} \right\} \right) \geqslant 1.$$

Numerical experiments: acoustics in 2D

- $\Omega = (0,1) \times (0,1)$
- $X = \{v \in H^1(\Omega) \; : \; v(0,x_2) = 0, x_2 \in (0,1)\}$
- $A(\mu) := -\partial_{x_1 x_1} - \mu_1 \partial_{x_2 x_2} - \mu_2$,
- $\mathcal{P} = [0.2, 1.2] \times [10, 50]$
- Neumann b.c. on top: $g_N = \cos(\pi x)$
- 





- high dimensional discretization: linear FE, $h = 0.01$ in each direction

Histograms of effectivity index $\widetilde{\Delta}(\mu)/\|u(\mu) - u^N(\mu)\|_{H^1(\Omega)}$

5 realizations

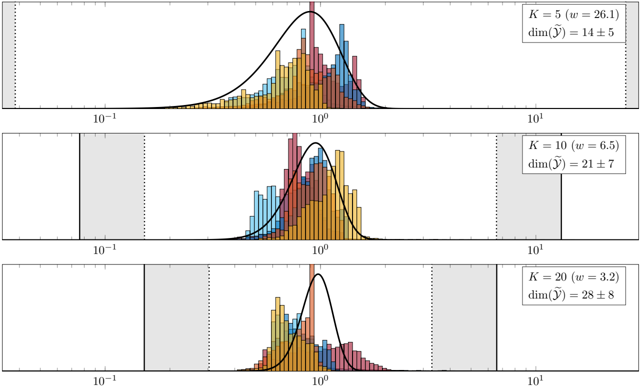


Figure: $\#\mathcal{S} = 10^4$, $\dim(X^N) = 20$, vertical dashed lines: $1/w$ and w, grey area: $1/(tol\, w)$ and $tol\, w$, where $\alpha \approx tol$, $tol = 2$, solid lines: chi-squared distribution

Histograms of effectivity index $\widetilde{\Delta}(\mu)/\|u(\mu) - u^N(\mu)\|_{H^1(\Omega)}$

100 realizations

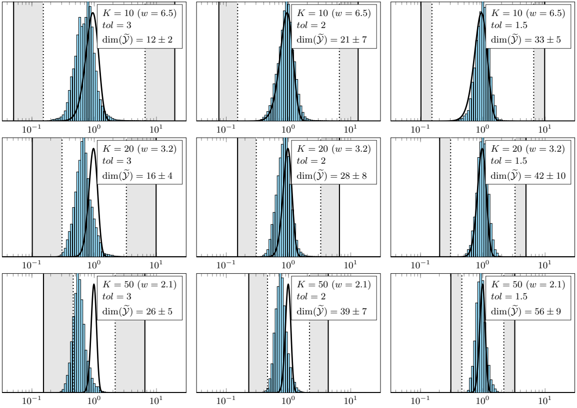


Figure: $\#\mathcal{S} = 10^4$, $\dim(X^N) = 20$, vertical dashed lines: $1/w$ and w, grey area: $1/(tol\, w)$ and $tol\, w$, where $\alpha \approx tol$, $tol = 2$, solid lines: chi-squared distribution

Summary

- Reduced (local approximation) spaces generated by methods from Randomized Linear Algebra
  - Probabilistic a priori error bound/Numerical experiments: convergence rate is only slightly worse compared to the optimal rate (factor $\sqrt{n}$).
  - Probabilistic a posteriori error bound allows to build the reduced space adaptively
  - required number of local solutions of PDE scale (roughly) with size of the reduced space; Numerical experiments: faster than Lanczos
- Proposed randomized a posteriori error estimator for projection-based model order reduction methods that...
  - ... is based on concentration inequalities, error-residual relationship, and random dual problem
  - ... does only contain computable constants
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  - ... has a favorable computational complexity as $\dim(\widetilde{\mathcal{Y}})$ can be chosen relatively small

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Thank you very much for your attention!