Structure preserving model reduction of network systems

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Reducing dimensions in Big Data: Model Order Reduction in action

Utrecht, 8 June 2018

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Outline

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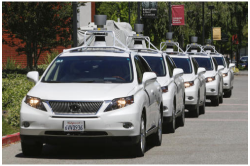
Background

Network models capture the behaviors and dynamics of many interconnected physical systems.

Power grid

Robots in our lab



Transportation network

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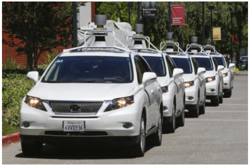
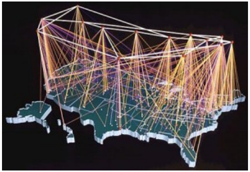


Transportation network

Network systems tend to be large-scale.



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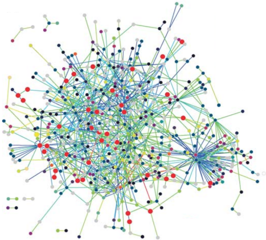
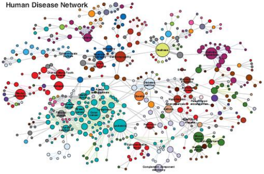


Biological networks (The protein interaction network of Treponema pallidum, Wikimedia commons)



Disease network (from CCSB)

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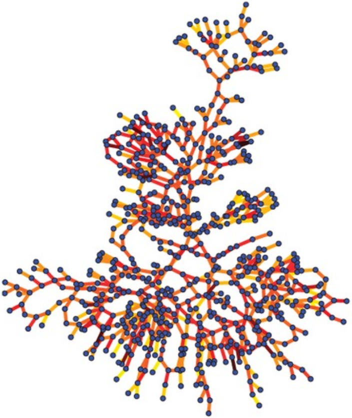
Power grid of Northern Italy [1]



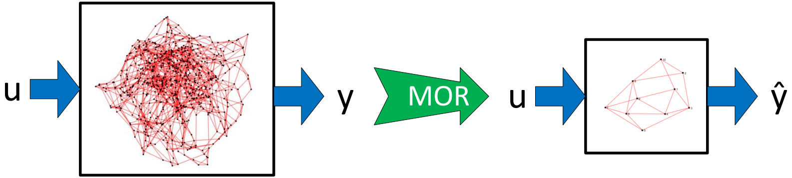
Representation of the physical network of transmission lines, which has 678 nodes and 822 edges.

Such large-scale network complicates analysis and synthesis.

Apart from the network structure, the physical structure important, i.e., 2nd order structure.

[1] Motter, Adilson E., et al. "Spontaneous synchrony in power-grid networks." Nature Physics 9.3 (2013): 191-197.

Problem setting



Model reduction with preservation of network structure

Systems with network structure have useful properties.

- Consensus and synchronization.
- Spatial structure is crucial for distributed controller design, sensor allocation, etc..
- Specific physical structure, such as second order structure, is useful.
- Reduced-order models ease analysis, controller design, etc.

Order reduction methods

Various existing model order reduction methods:

- Balanced realizations based order reduction.

  Linear: Moore '81, and many more, overview Antoulas '05, Nonlinear: S. '93, Fujimoto/S'10, and a few more.....

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Just applying these methods to networks does not preserve the network structure.

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Order reduction methods for graphs

Clustering based model order reduction:

- Recent interest, clustering of nodes.

  Ishizaki et al. '14, Monshizadeh et al.'14, Van der Schaft '14, Besselink et al."16

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Structure preservation for power systems reduction:

- Kron reduction.

  Van der Schaft'10, Dörfler/Bullo'13, Caliskan/Tabuada'14

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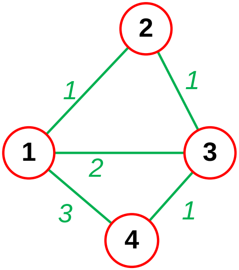
Various issues to deal with, topic of this lecture!

Graph theory

- An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where $|\mathcal{V}| = n$, $|\mathcal{E}| = m$.
  $\mathcal{V}$: a set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$: a set of edges, $\mathcal{W}$: the weights of edges
- The Laplacian matrix of the undirected graph $\mathcal{G}$ is L, defined by

$$L_{ij} = \begin{cases} \text{degree of node } i, & i = j \\ -w_{ij}, & i \neq j \text{ and node } i \text{ is adjacent to } j \\ 0, & \text{otherwise} \end{cases}$$



Laplacian matrix

$$L = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -1 & 2 & -1 & 0 \\ -2 & -1 & 4 & -1 \\ -3 & 0 & -1 & 4 \end{bmatrix}.$$

Network system

Consider a network of single integrator systems:

$$\Sigma : M\dot{x} = -Lx + Bu.$$

- states: $x \in \mathbb{R}^n$, inputs: $u \in \mathbb{R}^p$
- evolves as a weighted connected undirected graph $\mathcal{G}$

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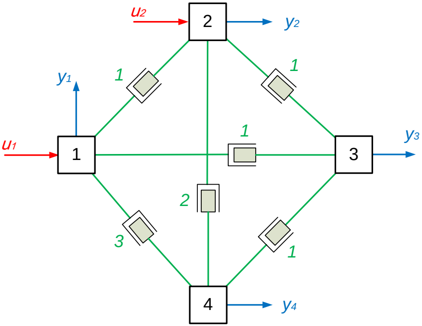
- evolves as a weighted connected undirected graph $\mathcal{G}$

- **Specific structure**:
  M is diagonal positive definite.
  L is a Laplacian matrix of $\mathcal{G}$.
- Semistable, NOT asymptotically stable

Example (mass-damper system)



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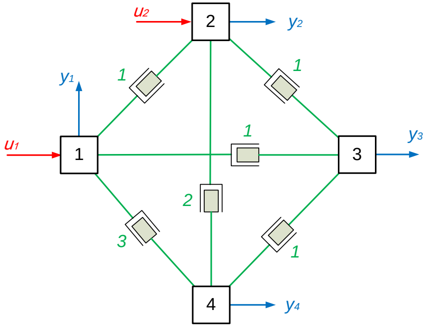

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = - \begin{bmatrix} 5 & -1 & -1 & -3 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ -3 & -2 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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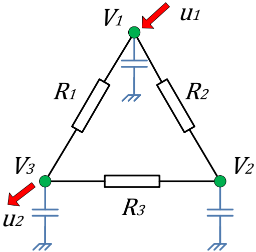
Example (electrical network)



- V_i: voltage of node i
- R_i: resistors
- u_i: current sources or sinks
- unit capacities

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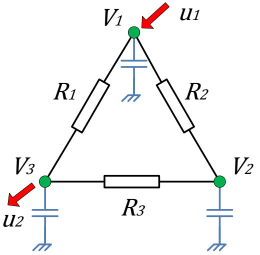

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Simple network system (Kirchhoff's voltage law):

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = -\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_1} & -\frac{1}{R_2} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_3} \\ \frac{1}{R_2} & -\frac{1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ 0 \\ -u_2 \end{bmatrix}$$

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Example (continued)

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Laplacian matrix (Kirchhoff matrix)

Similar to mass-damper system with external forces.

Example (continued)

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Similar to mass-damper system with external forces.

Network system

$$\mathbf{\Sigma} : \begin{cases} \dot x = -Lx + Bu, \\ y = Cx. \end{cases} \quad \text{with } x \in \mathbb{R}^n$$

Notice that Laplacian matrix L represents an undirected network, which satisfies $L \geq 0,\ L\mathbf{1} = 0$.

More general network systems

Results of this talk hold for a more general class of network systems (beyond single integrators), i.e., the dynamics of each agent is described by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bv_i \\ \theta_i &= Cx_i \end{aligned}$$

$x_i \in \mathbb{R}^{\bar{n}}$, $v_i \in \mathbb{R}^{\bar{m}}$, $\theta_i \in \mathbb{R}^{\bar{m}}$, with diffusive coupling rule

$$m_i v_i = -\Sigma_{j=1, j\neq I}^{n} w_{ij}(\theta_i - \theta_j) + \Sigma_{j=1}^{p} f_{ij} u_j$$

$m_i > 0$, $u_j \in \mathbb{R}^{\bar{m}}$, $j \in \{1, \ldots, p\}$

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$m_i > 0$, $u_j \in \mathbb{R}^{\bar{m}}$, $j \in \{1, \ldots, p\}$ results in network dynamics given by

$$(M \otimes I)\dot{x} = (M \otimes A - L \otimes BC)\, x + (F \otimes B) u$$

where the Laplacian L includes the w_{ij}, and $M =$diag$(m_1, \ldots, m_n) > 0$.
Semistable, NOT asymptotically stable.

Problem Formulation

Given a network system

$$\Sigma : (M \otimes I)\dot{x} = (M \otimes A - L \otimes BC)\,x + (F \otimes B)u.$$

with A no poles in the RHP, and the network system fulfills the synchronization property, i.e., $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$.

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Apply Petro-Galerkin projection with $W, V \in \mathbb{R}^{n\times r}$ $(r \ll n)$ full column rank, such that:

- $\hat{M} := W^T M V$ and $\hat{L} := W^T L V$ are again an inertia and Laplacian matrix, resp. Van der Schaft'14, Monshizadeh/Van der Schaft'14
- The reduced-order model $\hat{\mathbf{\Sigma}}$ has the same structure as above and $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|$ is small enough in terms of $\mathcal{H}_2$- or $\mathcal{H}_\infty$-norms.

Literature

- Monshizadeh et al. [1]
  Restrict to special clustering: Almost equitable partition
- Besselink et al. [2]
  Restrict to special topology: Tree graph
- Ishizaki et al. [3]
  Loss of the structure of Laplacian matrix

[1] N. Monshizadeh, H. L. Trentelman, and M. K. Camlibel, Projection- Based Model Reduction of Multi-Agent Systems Using Graph Partitions, IEEE Trans. Contr. Net. Syst., vol. 1, pp. 145-154, Jun. 2014.
A.J. van der Schaft, On model reduction of physical network systems, pp. 1419-1425 in Proc. MTNS2014, Groningen, the Netherlands, July 2014.
N. Monshizadeh, A.J. van der Schaft, Structure-preserving model reduction of physical network systems by clustering, Proc. 53rd IEEE CDC, Los Angeles, CA, USA, Dec. 2014.

[2] B. Besselink; H. Sandberg; K. Johansson, "Clustering-Based Model Reduction of Networked Passive Systems," in IEEE TAC, vol.61, no.10, 2016

[3] T. Ishizaki, K. Kashima, J. I. Imura, and K. Aihara, Model reduction and clusterization of large-scale bidirectional networks, IEEE TAC, vol. 59, pp. 48-63, 2014.

Clustering of graphs

Definition (Clustering)

node set $\mathcal{V} \Longrightarrow$ nonempty disjoint subsets $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r\}$.

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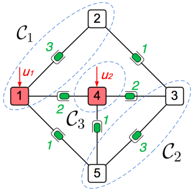
Consider a graph clustering $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r\}$ of $\mathcal{G}$ with n nodes. **Cluster matrix** $P \in \mathbb{R}^{n \times r}$:

$$P := [p(\mathcal{C}_1), p(\mathcal{C}_2), \cdots, p(\mathcal{C}_r)],$$

$p(\mathcal{C}_i) \in \mathbb{R}^n$: the k-th element is 1 if $k \in \mathcal{C}_i$, 0 otherwise.



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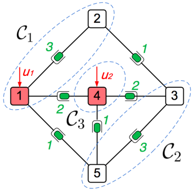
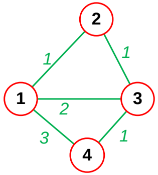


Take the cluster matrix for projection, i.e., $W = V = P$, then

$\hat{M} := P^T M P$ diagonal positive-definite;
$\hat{L} := P^T L P$ Laplacian of a connected undirected graph.

Example (structure preservation)



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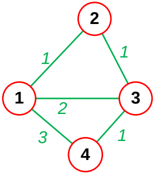

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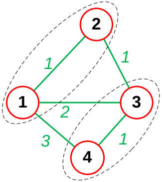

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\hat{L} = P^T L P = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}, \quad \hat{M} = P^T M P = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

Cluster selection

How to cluster, i.e.,

Problem

Find a clustering, with cluster matrix P such that the reduced-order model $\hat{\Sigma}$ approximates the original system Σ well.

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Aggregate nodes with similar behavior:

- How to capture behavior?
- How to quantify the differences in behavior?

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Aggregate nodes with similar behavior:

- How to capture behavior?
- How to quantify the differences in behavior?

Here we use

- input to state transfer functions of the nodes.
- and $\mathcal{H}_2$-norms for the difference.

Cluster selection

For general networks, under the assumptions on the A matrix and the synchronization property of the network, we have

$$\|\eta_i(s) - \eta_j(s)\|^2_{\mathcal{H}_2} = \int_0^\infty \mathbf{tr}\left[(e_i^T \otimes I)\xi(t)\xi^T(t)(e_j \otimes I)\right] dt,$$

with η_i the transfer function of node i, $\xi(t)$ the state impulse response of the system. It can be proven to be always bounded!

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(For a single integrator system:
$\|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2} = \|(e_i^T - e_j^T)(sM + L)^{-1}B\|_{\mathcal{H}_2}$)

Recursive clustering algorithm

Definition (Dissimilarity Matrix $\mathcal{D} \in \mathbb{R}^{n \times n}$)

$$\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2}$$

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Algorithm (recursive clustering)

- Step 1: Compute $\mathcal{D}$ matrix
- Step 2: Find the minimal off-diagonal entry $\mathcal{D}_{\mu\nu}$.
- Step 3: Cluster nodes μ and ν
- Step 4: Generate $\hat{\Sigma}$ by P resulting from the clustering in Step 3.
- Step 5: Repeat Step 1 to Step 4 until obtaining r-th order model.

Other clusterings (e.g., hierarchical) possible.

Illustrative example

mass-damper network system:

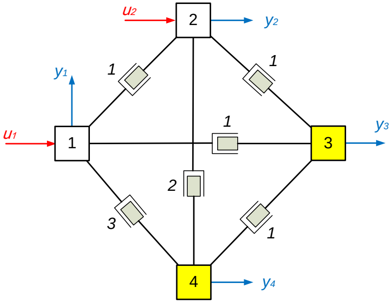

Figure: Topology of the original network

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

$$L = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 4 & -1 \\ -3 & -1 & -1 & 5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T, \ C = I_4.$$

Illustrative example (continued)

- Step 1: Compute the dissimilarity matrix

$$\mathcal{D} = \begin{bmatrix} 0 & 0.2940 & 0.1342 & 0.1472 \\ 0.2940 & 0 & 0.2098 & 0.2322 \\ 0.1342 & 0.2098 & 0 & 0.0277 \\ 0.1472 & 0.2322 & 0.0277 & 0 \end{bmatrix}.$$

- Step 2: The minimal value is 0.0277, which implies that the nodes 3 and 4 have most similar behaviors.

Illustrative example (continued)

- Step 1: Compute the dissimilarity matrix

$$\mathcal{D} = \begin{bmatrix} 0 & 0.2940 & 0.1342 & 0.1472 \\ 0.2940 & 0 & 0.2098 & 0.2322 \\ 0.1342 & 0.2098 & 0 & 0.0277 \\ 0.1472 & 0.2322 & 0.0277 & 0 \end{bmatrix}.$$

- Step 2: The minimal value is 0.0277, which implies that the nodes 3 and 4 have most similar behaviors.
- Step 3: Clustering: $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{2\}$, $\mathcal{C}_3 = \{3, 4\}$.
- Step 4: Projection matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Illustrative example (continued)

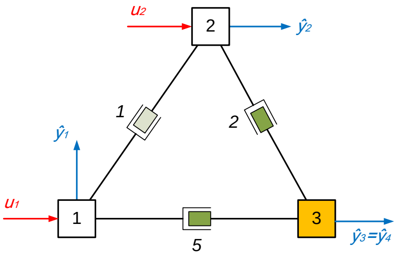

Figure: Topology of the reduced-order network

$$\hat{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

$$\hat{L} = \begin{bmatrix} 6 & -1 & -5 \\ -1 & 3 & -2 \\ -5 & -2 & 7 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T,$$

$$\hat{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Error bound

- An error bound expression for one step clustering in the single integrator case is easily computable.

Error bound

- An error bound expression for one step clustering in the single integrator case is easily computable.
- An error bound for the general network description reduced to r clusters is given by dissimilarity between clusters, i.e.,

$$\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2} \leq \gamma_a \sum_{k=1}^{r} \max_{i,j \in \mathcal{C}_k} \mathcal{D}_{ij},$$

where γ_a is a scalar satisfying an LMI, i.e.,

$$\begin{bmatrix} M \otimes (A^T + A) - L \otimes (C^T B^T + BC) & L \otimes BC & -I \\ L \otimes C^T B^T & -\gamma_a I & I \\ -I & I & -\gamma_a I \end{bmatrix} \prec 0$$

Illustrative example (continued)

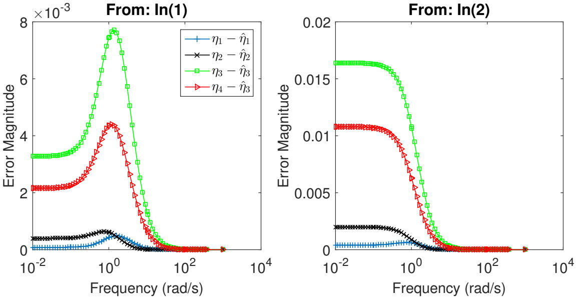

Figure: Plot of the error magnitude.

The error bounds are:

$$\|\eta_1(s) - \hat{\eta}_1(s)\| \leq 0.196, \; \|\eta_2(s) - \hat{\eta}_2(s)\| \leq 0.039,$$
$$\|\eta_3(s) - \hat{\eta}_3(s)\| \leq 0.046, \; \|\eta_4(s) - \hat{\eta}_3(s)\| \leq 0.037.$$

Properties of the reduced model

The reduced-order network system $\hat{\Sigma}$ has the following properties:

- The synchronization property is preserved, i.e., if the initial conditions are "equal" for the full order and reduced order network, then the nodes of the reduced order network converge to the same value as the nodes of the original network.

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- $\|\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}\|_{\mathcal{H}_2}$ and $\|\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}\|_{\mathcal{H}_\infty}$ are bounded.
- If nodes μ and ν are clustered
  Then, $\eta_i(s) - \hat{\eta}_i(s) = K_i(s)[\eta_\mu(s) - \eta_\nu(s)]$

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Minimal network realization

What is a minimal network realization?

Definition

A network realization is minimal if there are no 0-dissimilar vertices in the network, i.e., if $\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2} \neq 0$ for $i \neq j$.

Thus, if $D_{ij} = 0$, then node i and j are 0-dissimilar, and the network is non-minimal.

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Thus, if $D_{ij} = 0$, then node i and j are 0-dissimilar, and the network is non-minimal.

Relation with "conventional" minimality of the overall multi-agent system:

Relation

If Σ is minimal (controllable and observable), then Σ is also a minimal network realization.

Not vice versa!

Computational issues

Normally for calculating the $\mathcal{H}_2$ norm (necessary for the dissimilarity) the controllability Gramian can be used. However, the system is not asymptotically stable.

- Solution: Use a type of network controllability Gramian.

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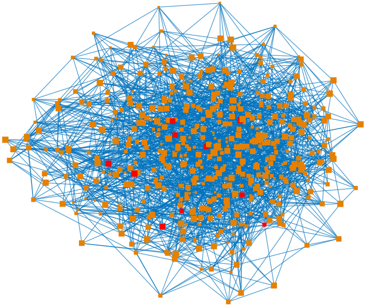
- Solution: Use a type of network controllability Gramian.

Iterative clustering and hierarchical clustering can be computationally expensive.

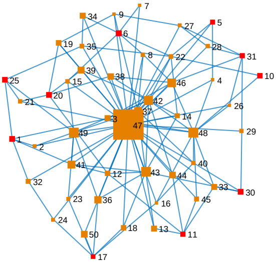
- Solution: Hierarchical clustering considers proximity of various clusters, and can be approximated by taking average dissimilarities (per cluster) of vertices to determine the proximity.

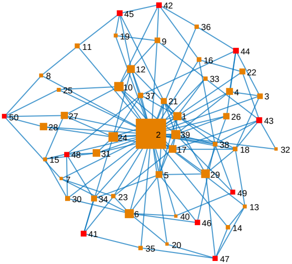
Small world network example



Left: Watts-Strogatz network with 500 nodes, 2000 edges, and 10 inputs.

Bottom: Approximated by iterative clustering (left) and hierarchical clustering (right) with 50 clusters.





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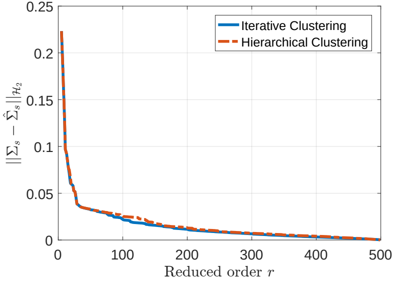
Small world network example continued

Table: Comparison of two algorithms for a reduced order of 50

Algorithm	$\|\boldsymbol{\Sigma}_s - \hat{\boldsymbol{\Sigma}}_s\|_{\mathcal{H}_2}$	Total computation time
Iterative clustering	0.0324	4142.5729s
(Appr.) hierarchical clustering	0.0330	38.5899s



Approximation error versus reduced order r.

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Second-order network systems

Mass-spring-damper network

$$M\ddot{x} + D\dot{x} + Lx = Fu. \quad (1)$$

Example: linearized coupled oscillator model in the power grid.

Second-order network systems

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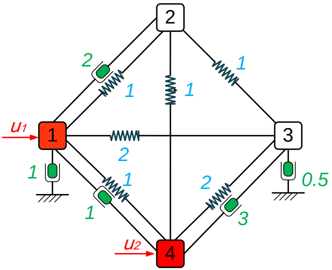
Example: linearized coupled oscillator model in the power grid.

Structural conditions

- M: diagonal positive definite;
- D: positive definite;
- L: Laplacian matrix of the weighted undirected graph describing the coupling among nodes.

A second-order network example



$$L = \begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 5 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix},$$

$$D = \begin{bmatrix} 4 & -2 & 0 & -1 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 3.5 & -3 \\ -1 & 0 & -3 & 4 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem formulation

- Given a second-order network system

$$\boldsymbol{\Sigma} : M\ddot{x} + D\dot{x} + Lx = Fu, \;\; y = x.$$

- Apply Petrov-Galerkin projection with the cluster matrix, i.e., $P \in \mathbb{R}^{n \times r}$ $(r \ll n)$

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- The reduced-order model

$$\hat{\mathbf{\Sigma}} : \hat{M}\ddot{z} + \hat{D}\dot{z} + \hat{L}z = P^T Fu,$$

with $\hat{M} := P^T MP$, $\hat{D} := P^T DP$, $\hat{L} := P^T LP$.

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with $\hat{M} := P^T M P$, $\hat{D} := P^T D P$, $\hat{L} := P^T L P$.

Find a suitable clustering such that $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|$ is small enough in terms of $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms

Existing methods

"Conventional" second-order model reduction methods

- Second-order balanced truncation

  Meyer & Srinivasan'96, Reis & Stykel, Benner & Saak, Losse & Mehrmann, Y. Chahlaoui, et al.'06. .....

- Krylov subspace methods

  Salimbahrami & Lohmann, Bai & Su'05, Su & Craig Jr. ...

Limitation: network structure is not preserved.

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Limitation: network structure is not preserved.

Clustering-based approach

- Ishizaki & Imura '15

Limitation:

- Loss of the structure of Laplacian matrix
- The system has to be asymptotically stable

Network Gramian

Again: dissimilarity matrix defined and relevant for clustering!
For computation, the network Gramian is relevant, i.e.,

Network controllability Gramian

$$\mathcal{P} = \int_0^\infty (e^{\mathcal{A}\tau} - \mathcal{J})\mathcal{B}\mathcal{B}^T(e^{\mathcal{A}^T\tau} - \mathcal{J}^T)d\tau.$$

where $\mathcal{A} = \begin{bmatrix} \mathbf{0} & I \\ -M^{-1}L & -M^{-1}D \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} \mathbf{0} \\ M^{-1}F \end{bmatrix}$, and $\mathcal{J} = \lim_{\tau\to\infty} e^{\mathcal{A}\tau} = \frac{1}{\mathbf{1}^T D \mathbf{1}} \begin{bmatrix} \mathbf{1}\mathbf{1}^T D & \mathbf{1}\mathbf{1}^T M \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}.$

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$\mathcal{P}$ fulfills a Lyapunov type of equation, and the rank has a direct relation with controllability of the original semi-stable system!

Network Gramian

The network Gramian $\mathcal{P}$ is useful for fast computation of dissimilarity and error bound computation, i.e., for dissimilarity:

$$\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2} = \sqrt{H\mathcal{P}H^T}, \text{ with } H = [(e_i - e_j)^T, \mathbf{0}_{1\times n}]$$

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Error bound

$$\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_2} = \sqrt{\mathbf{tr}\left(\mathcal{C}_{\mathbf{e}} \begin{bmatrix} \mathcal{P}_n & \mathcal{P}_\text{x} \\ \mathcal{P}_\text{x}^T & \mathcal{P}_r \end{bmatrix} \mathcal{C}_{\mathbf{e}}^T\right)},$$

$\mathcal{P}_n$ is the network controllability Gramian of the original network
$\mathcal{P}_r$ is of the reduced order network
$\mathcal{P}_\text{x}$ is a coupling term that is a solution to a Sylvester like equation.
$\mathcal{C}_{\mathbf{e}}$ is the output matrix of the error system.

Illustrative example

Mass-damper-spring system:

$$\mathcal{D} = \begin{bmatrix} 0 & 0.1920 & 0.2449 & 0.2424 \\ 0.1920 & 0 & 0.0940 & 0.1163 \\ 0.2449 & 0.0940 & 0 & 0.0333 \\ 0.2424 & 0.1163 & 0.0333 & 0 \end{bmatrix}.$$

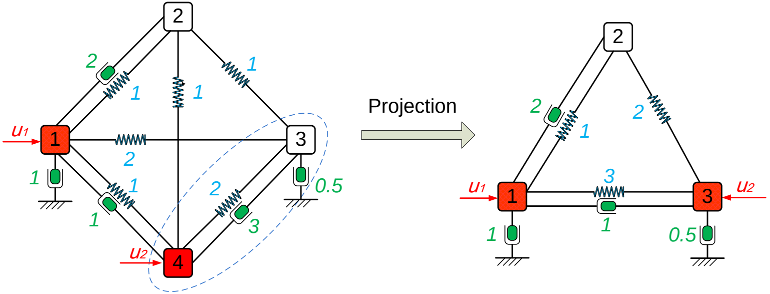
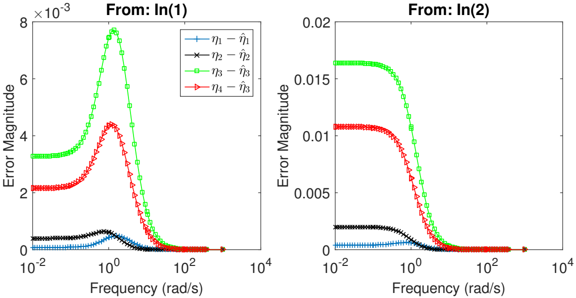


Figure: Illustration of second-order network system

Illustrative example (continued)



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Some remarks

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- Relation with clustering as above, and network minimality not yet clear. Future work!

Background on balancing

- Balancing based on diagonalizing two Gramians. For asymptotically stable systems the controllability and observability Gramian.

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- Balancing based on diagonalizing two Gramians. For asymptotically stable systems the controllability and observability Gramian.
- Balanced truncation removes badly controllable and badly observable states, hence takes control system considerations into account.
- Balanced truncation preserves stability and balanced structure. However, other structure (such as network structure!) is generally not preserved.
- Network systems are semi-stable, thus balanced truncation not directly applicable. And how about the network structure?

  $\rightarrow$ Enforce network structure, and separate marginally stable part from asymptotically stable part.

- Reduce the system, possibly first destroying network structure.
- Make sure the reduced system can be transformed into a network system, i.e., when can a linear system be transformed into a network system through a state transformation?

$$\tilde{\Sigma} : \begin{cases} \dot{\hat{z}} = \mathcal{A}\hat{z} + \mathcal{B}u, \\ \hat{y} = \mathcal{C}\hat{z}, \end{cases} \quad \overset{\hat{z} = T\hat{x}}{\Longleftrightarrow} \quad \hat{\Sigma} : \begin{cases} \dot{\hat{x}} = -\hat{L}\hat{x} + \hat{F}u, \\ \hat{y} = \hat{H}\hat{x}, \end{cases}$$

Network structure preservation

- Reduce the system, possibly first destroying network structure.
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Questions:

A. Which property of $\tilde{\Sigma}$ implies there exists a T s.t. $\hat{L}$ is a Laplacian?

B. How to ensure the system after balanced truncation has this property?

Network structure preservation

Focus on single integrator systems:

$$\tilde{\Sigma} : \begin{cases} \dot{\hat{z}} = \mathcal{A}\hat{z} + \mathcal{B}u, \\ \hat{y} = \mathcal{C}\hat{z}, \end{cases} \quad \overset{\hat{z} = T\hat{x}}{\Longleftrightarrow} \quad \hat{\Sigma} : \begin{cases} \dot{\hat{x}} = -\hat{L}\hat{x} + \hat{F}u, \\ \hat{y} = \hat{H}\hat{x}, \end{cases}$$

Theorem

$\tilde{\Sigma}$ is equivalent to a network system $\hat{\Sigma}$ (or $\mathcal{A}$ is similar to $-\hat{L}$) **if and only if** all the eigenvalues of $\mathcal{A}$ are real, and one of its eigenvalues is zero, i.e.

$$\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_{n-1}(A) < \lambda_n(A) = 0$$

Splitting up the system

First separate the zero eigenvalue of the Laplacian:

$$\Sigma : \begin{cases} \dot{x} = -Lx + Fu, \\ y = Hx. \end{cases}$$

Semistable System

Transform the Laplacian:

$$\begin{aligned} L &= T\Lambda_L T^T \\ &= \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_L & \\ & 0 \end{bmatrix} \begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix}, \end{aligned}$$

$$T_1 \in \mathbb{R}^{n\times(n-1)},\ T_2 = \frac{\mathbf{1}}{\sqrt{n}}$$

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Splitting up the system

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$$\boldsymbol{\Sigma} : \begin{cases} \dot{x} = -Lx + Fu, \\ y = Hx. \end{cases}$$

Semistable System

$$\boldsymbol{z_s = T_1 x}$$
$$\boldsymbol{z_a = T_2 x}$$

$$\boldsymbol{\Sigma_s} : \begin{cases} \dot{z}_s = Az_s + Bu, \\ y_s = Cz_s, \end{cases}$$
where $A = A^T = -\bar{\Lambda}_L$,

Stable System

Transform the Laplacian:

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$$= \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_L & \\ & 0 \end{bmatrix} \begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix},$$

$$T_1 \in \mathbb{R}^{n\times(n-1)}, \; T_2 = \frac{\mathbf{1}}{\sqrt{n}}$$

$$\boldsymbol{\Sigma_a} : \begin{cases} \dot{z}_a = \frac{1}{\sqrt{n}}\mathbf{1}^T Fu, \\ y_a = \frac{1}{\sqrt{n}} H\mathbf{1} z_a. \end{cases}$$

Average System

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Generalized Gramians

Consider the stable subsystem:

$$\boldsymbol{\Sigma_s} : \begin{cases} \dot{z}_s = Az_s + Bu, \\ y_s = Cz_s, \end{cases} \text{ where } z_s \in \mathbb{R}^{n-1}$$

Generalized Controllability Gramian P and Observability Gramian Q

$$\min \operatorname{trace}(P)$$
$$AP + PA + BB^T \leq 0,$$

$$\min \operatorname{trace}(Q)$$
$$Q \text{ is diagonal;}$$
$$A^T Q + QA + C^T C \leq 0.$$

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Q is diagonal;
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Balancing Transformation such that in new coordinates:

$$TPT^T = T^{-T} Q T^{-1} = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_{n-1}),$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n-1}$ are generalized Hankel singular values.

Generalized balanced truncation

- **Truncation:** If $\sigma_r \gg \sigma_{r+1}$, then truncation can be applied, i.e., put $x_{r+1} = \cdots = x_{n-1} = 0$.

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Result

All the eigenvalues of the reduced order system matrix $\hat{A}$ are real negative. Thus $\begin{bmatrix} \hat{A} & \\ & 0 \end{bmatrix}$ is similar to a Laplacian matrix, and thus there exists a transformation s.t. the reduced system is a network system.

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The standard error bound now holds:

$$\|\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}\|_\infty = \|\boldsymbol{\Sigma_s} - \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{s}}\|_\infty \leq 2 \sum_{i=r}^{n-1} \sigma_i,$$

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Illustrative Example

For comparison, we consider an network example in Monshizadeh et al. '14 and Mlinarić et al.'15:

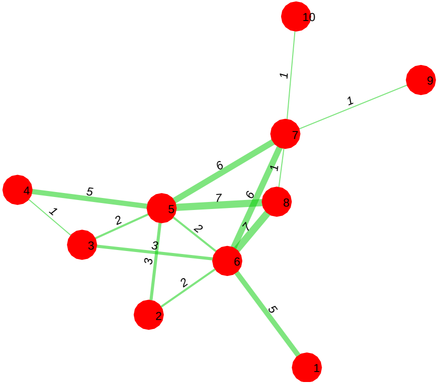


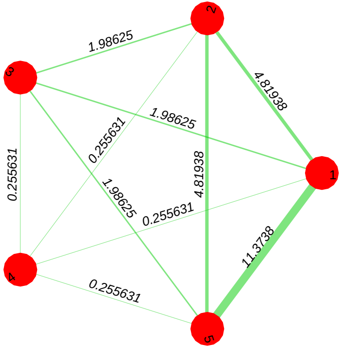
Figure: A network example with 10 vertices where the edge weights are shown by the thickness of the connection lines

Illustrative Example

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -1 & -2 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 6 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 & 25 & -2 & -6 & -7 & 0 & 0 \\ -5 & -2 & -3 & 0 & -2 & 25 & -6 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & -6 & 15 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -7 & -7 & -1 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{L} = \begin{bmatrix} 18.4350 & -4.8194 & -1.9862 & -0.2556 & -11.3738 \\ -4.8194 & 11.8806 & -1.9862 & -0.2556 & -4.8194 \\ -1.9862 & -1.9862 & 6.2144 & -0.2556 & -1.9862 \\ -0.2556 & -0.2556 & -0.2556 & 1.0225 & -0.2556 \\ -11.3738 & -4.8194 & -1.9862 & -0.2556 & 18.4350 \end{bmatrix}$$

Illustrative Example



Approaches	Proposed Method	Monshizadeh	Mlinarić
$\dfrac{\Vert\Sigma - \hat{\Sigma}\Vert_{\mathcal{H}_2}}{\Vert\Sigma\Vert_{\mathcal{H}_2}}$	0.0849	0.5270	0.1459

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- This balancing based model reduction procedure always allows to obtain a network system.

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- This balancing based model reduction procedure always allows to obtain a network system.
- However, there is only a guarantee that the system can be represented by a complete graph, other structures cannot be guaranteed.
- Second order structure is not necessarily preserved either, potentially gradient system approach can help. S. /Van der Schaft'11
- More general network structure is also possible, combined passivity preserving balancing and balancing of the network. In fact, nodes and Laplacian reduction can be separated, Cheng/S./Besselink'17.

Some of my references on networks

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Conclusion

Summarizing:

- Structure preserving order reduction methods with error bounds based on clustering based on dissimilarity for first and second order networks.
- A generalized balancing approach that preserves the network structure is introduced.

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- Structure preserving order reduction methods with error bounds based on clustering based on dissimilarity for first and second order networks.
- A generalized balancing approach that preserves the network structure is introduced.

Various new concepts, such as network minimality, and a network Gramian are introduced. These are control systems concepts necessary for taking the reduction of networks a step further.

Outlook

- Directed graphs, Cheng/S. CDC17.
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- Directed graphs, Cheng/S. CDC17.
- Lur'e systems, Cheng/S ECC'18. Nonlinear networks? Differential balancing? Kawano/S.'17
- Time scale separation in networks, very relevant for e.g., power networks. Slow coherency and singular perturbation methods should be incorporated.
- Other energy applications such as the different time scales of the power and gas networks and market structures? .

Acknowledgement