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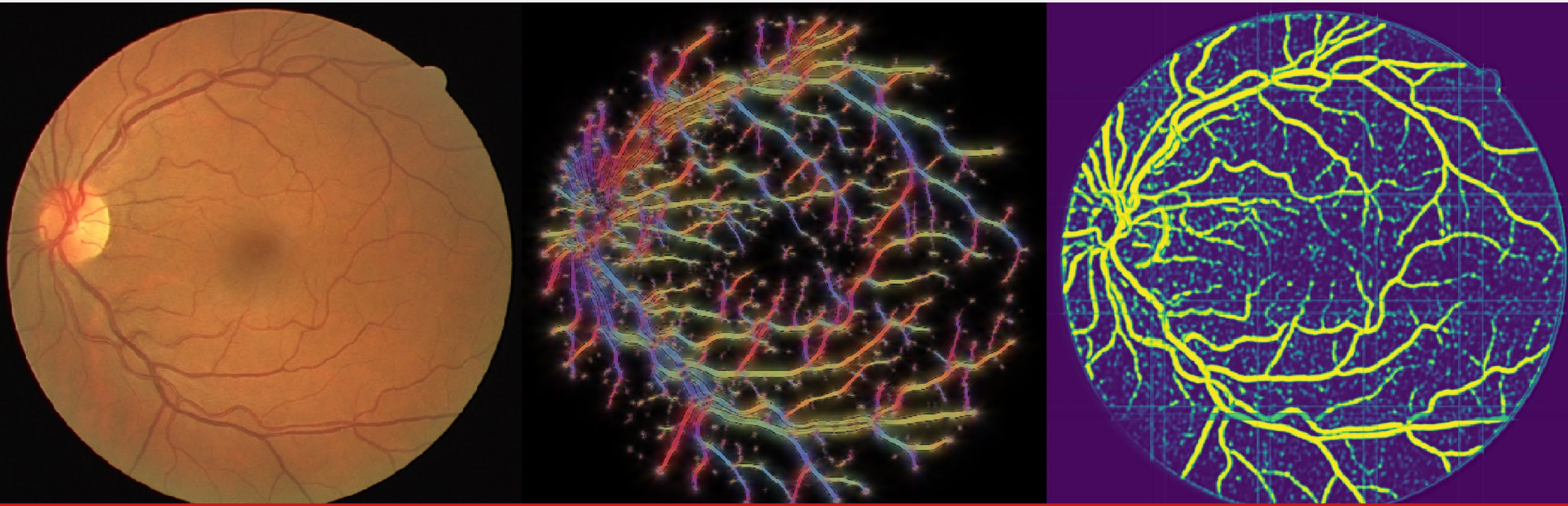

PDE-based CNNs

with Morphological Convolutions

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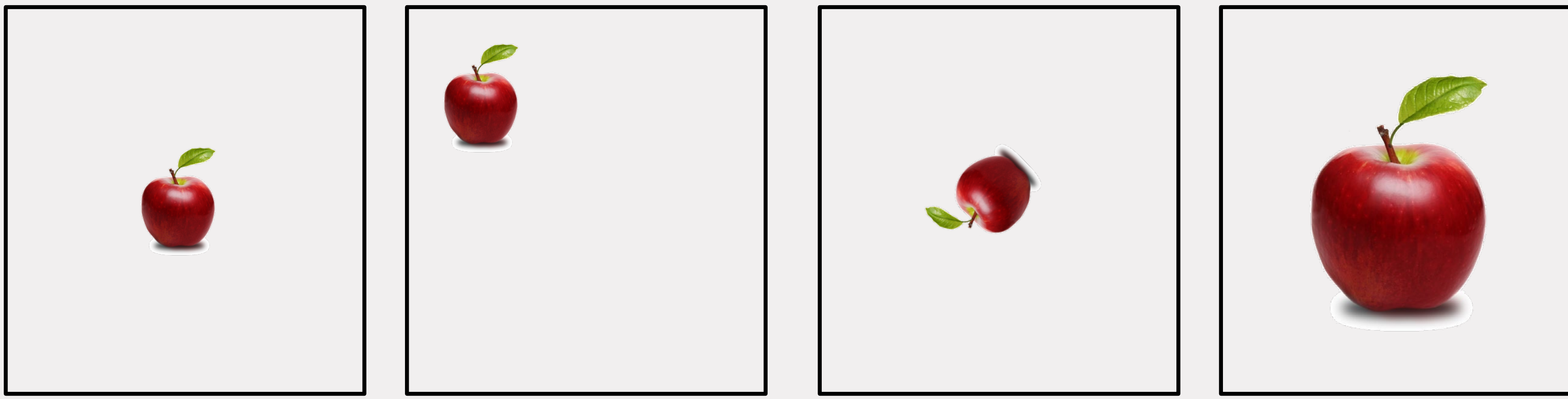
Basic Example

Modeling a heat process by a single layer CNN

Heat equation	$\equiv$	Analytic solution	$\equiv$	Single layer CNN
$\begin{cases} \frac{\partial f}{\partial t} = \nabla^2 f \\ f(0) = f_0 \end{cases}$		$f(t) = G_t * f_0$ with G_t the heat kernel for time t		$f_{out} = \sigma(K * f_0)$ With σ a ReLU and K a learnable kernel

Equivariance

Group CNNs

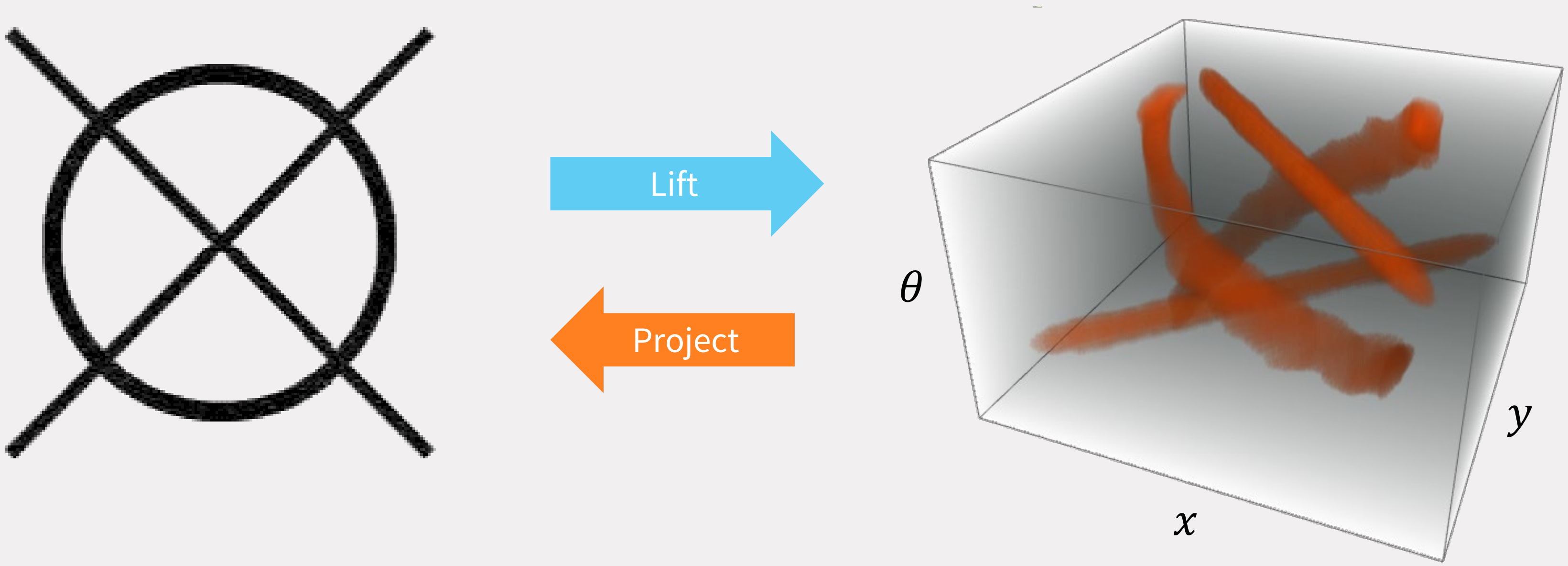


"Apple" "Apple" "Dog" "Boat"

	Translation	Rotation	Scaling
Spatial CNN	✓		
SE(d) CNN	✓	✓	
SIM(d) CNN	✓	✓	✓

Extending the domain

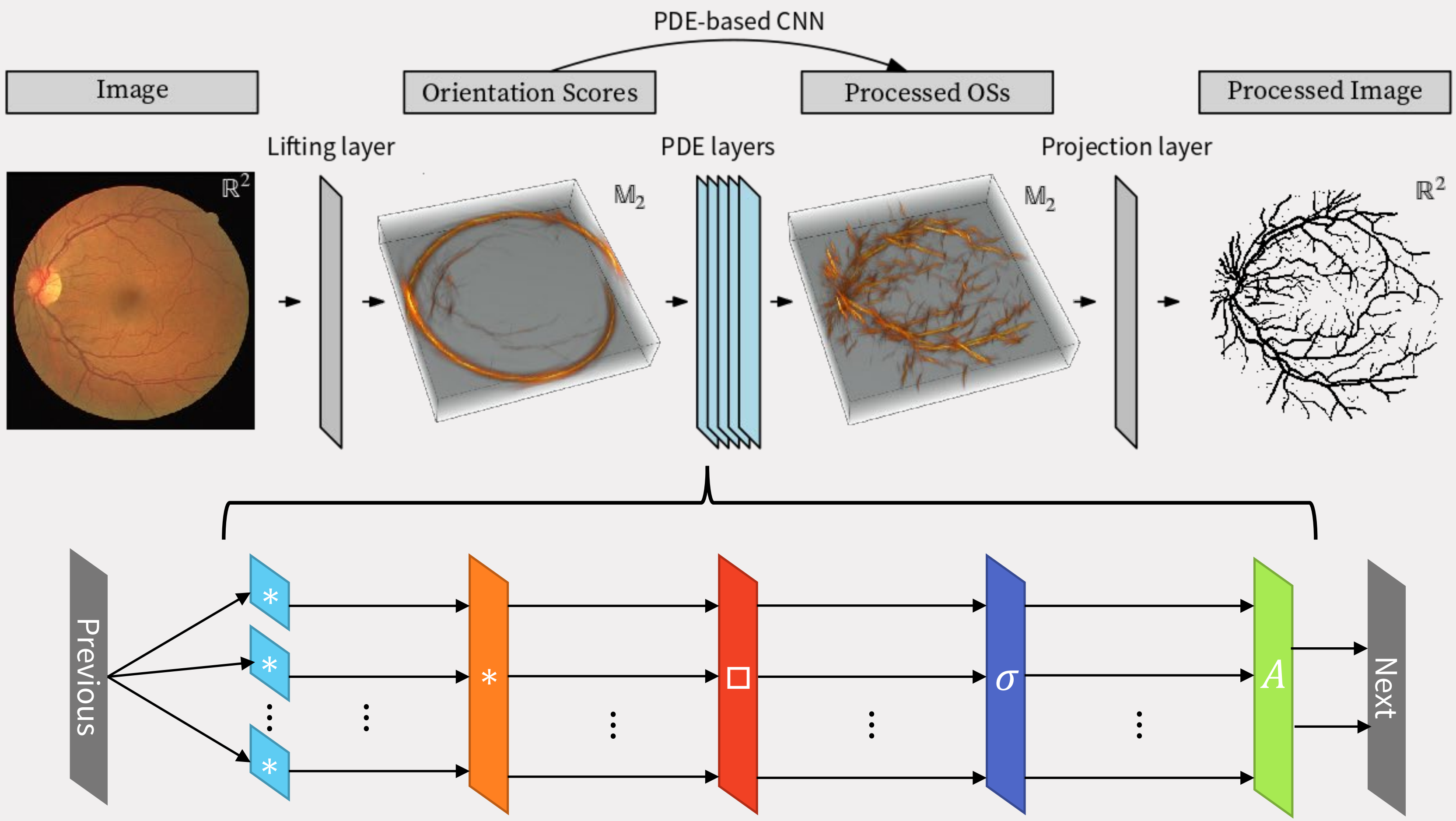
To a homogeneous space of the symmetry group



- Orientation Score Transform
- Can be learned
- More straightforward to design roto-translation equivariant CNNs

PDE-based CNN

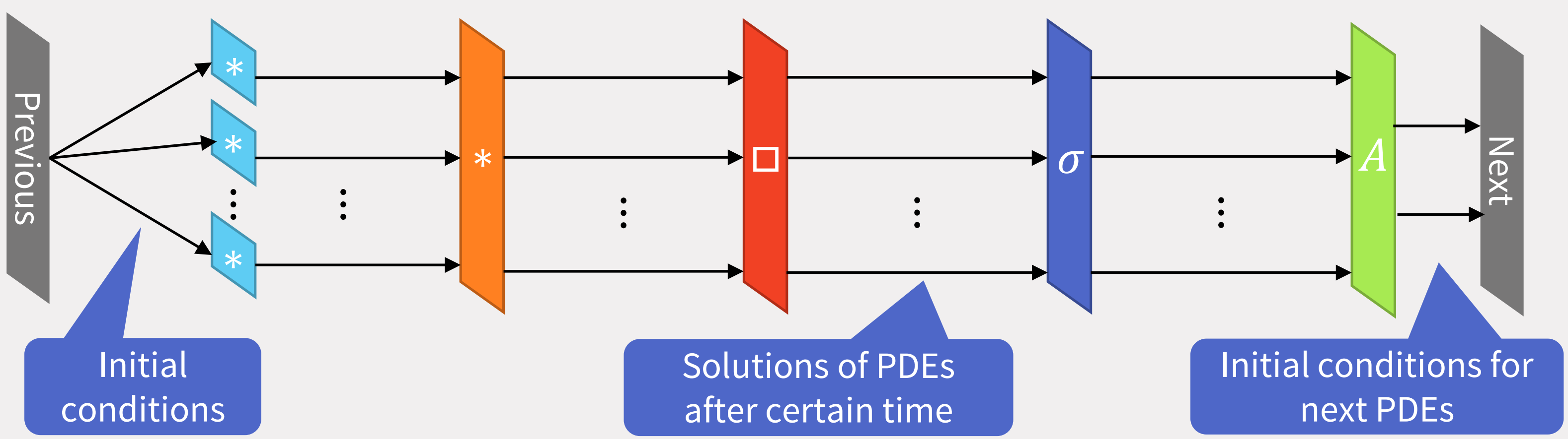
An example segmentation network



PDE Layer

$$\left(\frac{\partial W_k}{\partial t} = -\boldsymbol{c}_k \cdot \nabla W_k - |\Delta|^{\alpha} W_k \pm \|\nabla W_k\|^{2\alpha} \quad , W_k(\cdot, 0) = U_k(\cdot)\right)_{k=1}^{K}$$

Goal	**Transport**	**Regularization**	**Max pooling**	**Normalization**	**Combination**
PDE	Convection	Fractional diffusion	Dilation Erosion	Codomain transformation	Create initial conditions for next set of PDEs
Numerical operation	Convolution		Morph. Convolution	ReLU	Linear combinations



Parameters

What will we be training?

$$\frac{\partial W_k}{\partial t} = -\boldsymbol{c}_k \cdot \nabla_{\mathrm{G}_1} W_k - \left|\Delta_{G_2}\right|^{\alpha} W_k \pm \left\|\nabla_{\mathrm{G}_3} W_k\right\|_{G_3}^{2\alpha}$$

- K convection vectors
- 3 metric tensor fields
  - Inducing metrics d_{G_i} on the homogeneous space

- Design parameter: $\alpha \in \left(\frac{1}{2}, 1\right]$

Constructing Solutions

Let G be a Lie group (e.g. $SE(d)$)
Let G/H be a homogeneous space (e.g. $SE(d)/(\{\mathbf{0}\} \times SO(d-1))$)
G acts on G/H by $\odot$.
Let $f \in L^2, K_1 \in L^1(G/H, \mathbb{R})$.

- Linear Convolution

$$(K_1 * f)(x) = \int_G K_1(h^{-1} \odot x)\, f(h \odot e)\, d\mu(h)$$

- Morphological Convolution

$$(K_2 \square f)(x) = \sup_{h \in G}/\inf_{h \in G} \quad K_2(h^{-1} \odot x) + f(h \odot e)$$

  - sup : dilation
  - inf : erosion

Max Pooling

Morphological Convolution generalizes Max Pooling

For $\alpha \downarrow {}^1/_2$ the solution kernel to the morphological part converges to

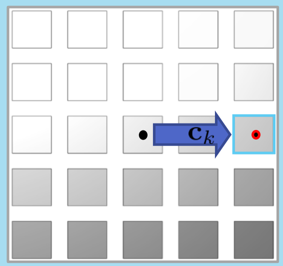
$$K_t(x) = \begin{cases} 0 & \text{if } \; d_{G_3}(e,x) \leq t, \\ -\infty & \text{else.} \end{cases}$$

$$\begin{aligned} (K_t \square f)(x) &= \sup_{h \in G} K(h^{-1} \odot x) + f(h \odot e) \\ &= \sup\left\{ f(h \odot e) \,\middle|\, h \in G : d_{G_3}(e, h^{-1} \odot x) \leq t \right\} \\ &= \sup\left\{ f(h \odot e) \,\middle|\, h \in G : d_{G_3}(h \odot e, x) \leq t \right\} \\ &= \sup_{y \in B(x,t)} f(y) \end{aligned}$$

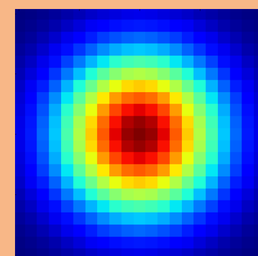
For $\alpha > {}^1/_2$: “soft” max pooling

Geometric Interpretation

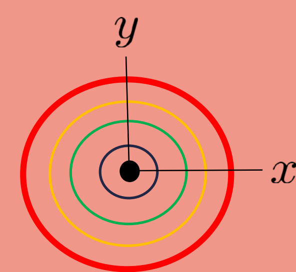
Transport



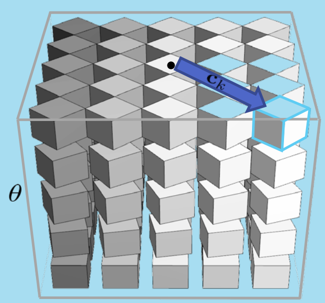
Regularization

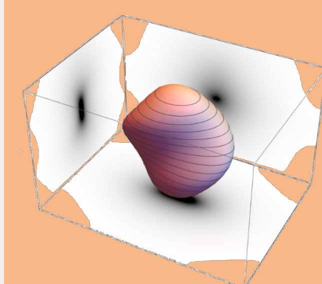


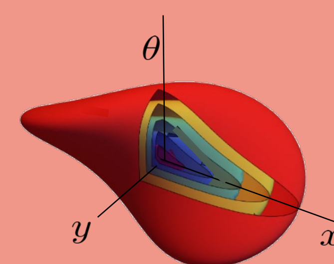
Max Pooling



$$\frac{\partial W_k}{\partial t} = -\boldsymbol{c}_k \cdot \nabla_{G_1} W_k \; - \left|\Delta_{G_2}\right|^{\alpha} W_k \; \pm \left\|\nabla_{G_3} W_k\right\|_{G_3}^{2\alpha}$$







Maintaining Equivariance

Conditions on the kernels?

- Linear convolution

$$\forall g \in G: \quad (K * f)(g \odot x) = \Big(K * \big(y \mapsto f(g \odot y)\big)\Big)(x)$$

$$\Leftrightarrow$$

$$\forall g, h \in G \;\; \forall x \in G/H: \quad K(hg \odot x) = K(gh \odot x)$$

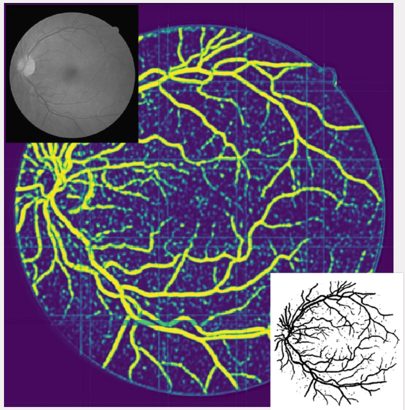
- Same for morphological convolution

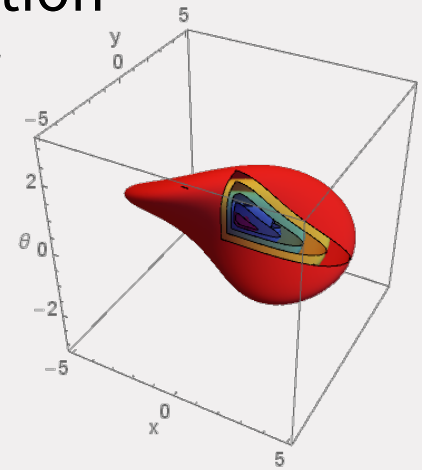
- Kernel symmetries are required!

First Experiment

Adding morphological convolution to a retinal segmentation network

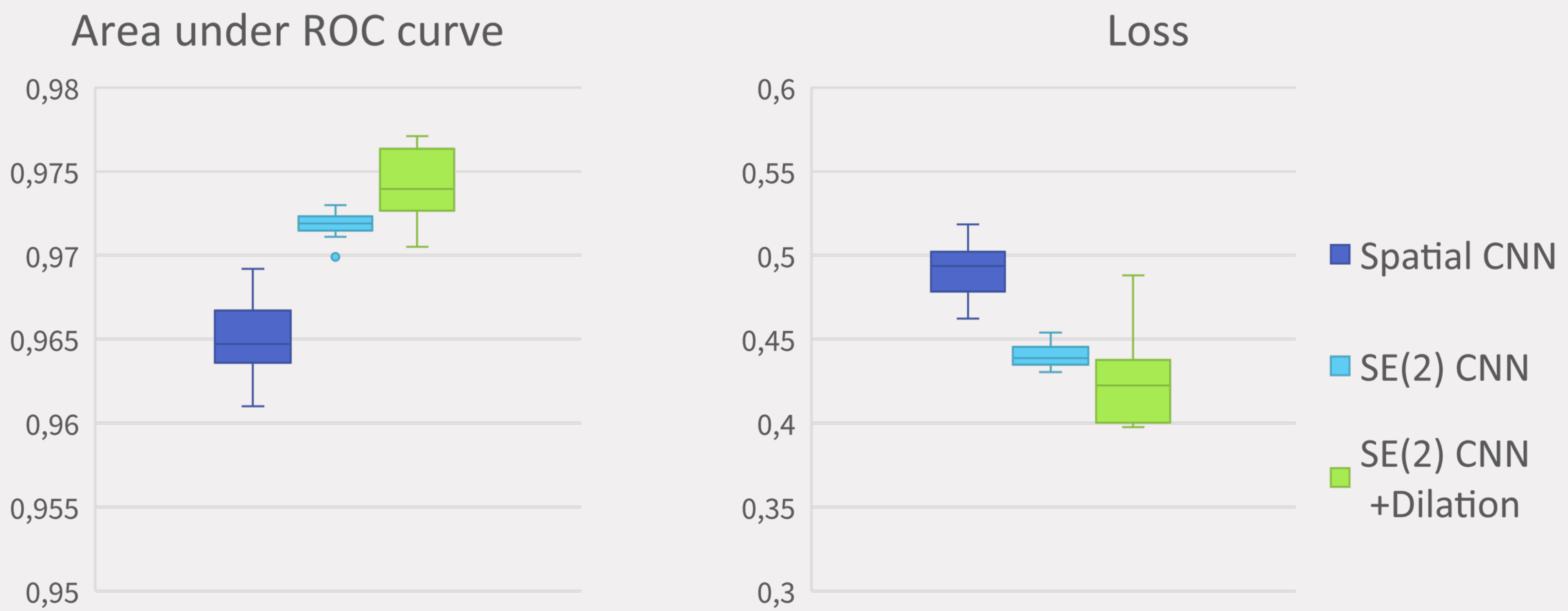
- Spatial CNN
  - 6 convolution layers
  - 34580 parameters
- SE(2) group CNN
  - Lift layer, 4 convolution layers, projection layer
  - 33916 parameters
- SE(2) group CNN with single morph. convolution
  - Lift layer, 4 convolution layers, dilation layer, projection layer
  - 33916 parameters
  - Fixed morph. convolution kernel for $\alpha = {}^2/_3$





First Experiment

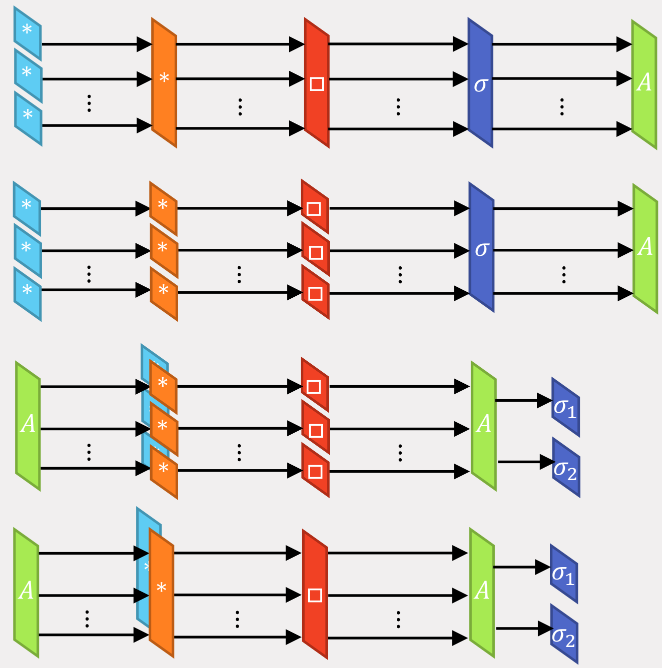
Performance improvement



TensorFlow

Current and Future Work

- TensorFlow implementation
  - experiments
- Layer architectures
- Geometric interpretability
- Probabilistic interpretability
- Integrate PDE framework for geometric equivariant processing of orientation scores (2005-now)

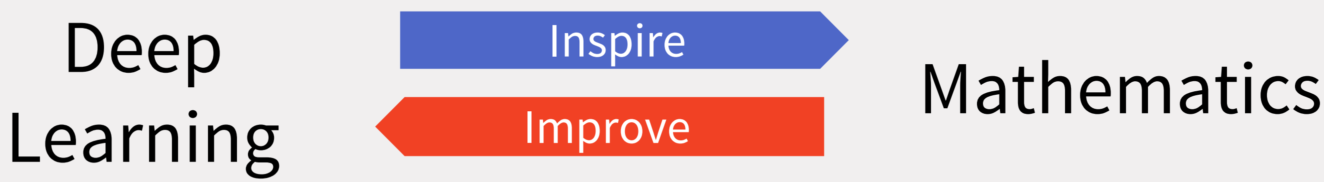


$\nabla \quad \Delta \quad \mathrm{div} \quad (\cdot,\cdot) \quad \|\cdot\|$

Concluding remarks

Geometric PDE framework for CNNs

- Improved performance over state-of-the-art G-CNNs
  - by inclusion of single PDE-based morphological convolution layer
- Problem symmetry integral part of the design



- Also see 2 talks in IHP Paris YouTube search: “Remco Duits IHP”
- Chapter 2 of my MSc thesis https://bmnsmets.com/publication/smets2019msc/