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Bounding Performance of Random Walks in the Positive Orthant

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Stochastic Operations Research, University of Twente

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- Phd from TU Delft, electrical engineering, 2010
  - information theory for wireless communication networks
  - in part at Ecole Polytechnique Fédérale de Lausanne, Switzerland
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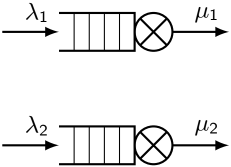
This talk is based on joint work with Xinwei Bai, Richard Boucherie, Yanting Chen, Tom Coenen and Jan-Kees van Ommeren.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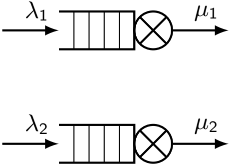
μ_1
λ_2
μ_2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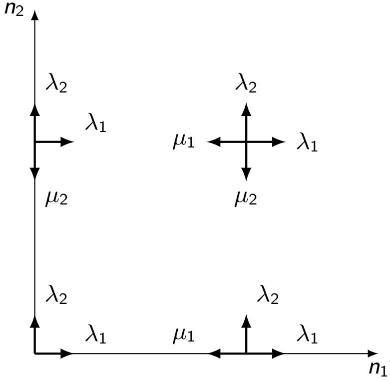
n_2
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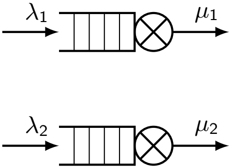
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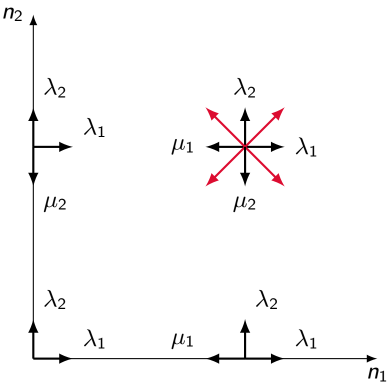
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Model: random walk in the positive orthant

- Discrete-time Markov chain on state space: $S = \{0, 1, \dots\}^D$
- Partition of S into components $C_1, C_2, \dots, C_K$ of the form

$$C_k = \prod_{d=1}^{D} \{b_\ell(k,d), \dots, b_u(k,d)\}.$$

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$$C_k = \prod_{d=1}^{D} \{b_\ell(k,d), \dots, b_u(k,d)\}.$$

- $p_u(n)$: Probability to jump from n to $n + u$
- Translation invariant transition probabilities in each component, i.e.,

$$\text{if } n, m \in C_k, \text{then } p_u(n) = p_u(m).$$

- Transitions to neighbours only

$$p_u(n) > 0, \text{ only if } u \in N = \{-1, 0, 1\}^D.$$

Problem Statement

- Irreducible, aperiodic and positive recurrent Markov chain
- Stationary probability distribution, $\pi : S \to [0, \infty)$, is unique solution to balance equations $\pi(n) = \sum_m \pi(m) p_{n-m}(m)$.
- Performance measure
  - $\mathcal{F} = \mathbb{E}_\pi[F]$, where $F : S \to [0, \infty)$.
  - Component-wise linear.
  - Examples: First moments, tail probabilities, blocking probabilities, ...

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Goal

Bound $\mathcal{F}$.

Stationary Distribution of Random Walks

- Boundary value problem for quarter plane / two dimensions:
  - Find generating function of invariant measure
  - Cohen & Boxma, 1983
  - Fayolle, Iasnogorodski & Malyshev, 1999
  - Depends on solution of conformal mapping
  - No generalization to more than two dimensions
- Matrix geometric approach:
  - Neuts, 1981
  - Latouche & Ramaswami, 1999
  - Depends on solution of non-linear matrix equation
  - No generalizations to more than two dimensions that preserve structure
- More than four dimensions:
  - Gamarnik, 2002: Positive recurrence is algorithmically undecidable

Geometric Product-form Random Walks

- In special cases we have

$$\pi(n) = \pi(n_1, \ldots, n_D) = \prod_{d=1}^{D} (1 - \rho_d)\rho^{n_d},$$

  $\rho_d \in (0, 1)$.
- Examples:
  - Jackson queueing networks (routing between independent servers)
  - Networks with negative customers
  - ...
- Characterization of product-form queueing networks
  - van Dijk "Queueing Networks and Product Forms: a system's approach", 1993
  - Latouche & Miyazawa, QUESTA, 2014

Approach to Bounding $\mathcal{F}$

- Construct a second/perturbed random walk with:
  - transition probabilities $\bar{p}_u(n)$.
  - known stationary distribution $\bar{\pi}$.
- Question: Can we approximate $\mathcal{F}$ in terms of $\bar{\pi}$?

$$\bar{p}_u(n) \approx p_u(n) \overset{??}{\Longleftrightarrow} \mathcal{F} \approx \sum_n \bar{\pi}(n) F(n)$$

A Bilinear Programming Approach to Error Bounds

Markov Reward Error Bound

- Interpret F as one-step reward function.
- Expected cumulative reward starting from state n:

$$F^t(n) = F(n) + \sum_u p_u(n) F^{t-1}(n+u),$$

for $t > 0$ and $F^0(n) = 0$.

- Bias term:

$$D_u^t(n) = F^t(n+u) - F^t(n).$$

Markov Reward Error Bound

$$F^t(n) = F(n) + \sum_u p_u(n) F^{t-1}(n+u),$$
$$D_u^t(n) = F^t(n+u) - F^t(n).$$

Theorem (van Dijk)

Let $\bar{F} : S \to [0, \infty)$ *and* $G : S \to [0, \infty)$ *satisfy*

$$\left| \bar{F}(n) - F(n) + \sum_u \left[\bar{p}_u(n) - p_u(n) \right] D_u^t(n) \right| \le G(n)$$

for all $n \in S$ *and* $t \geq 0$*. Then*

$$\left| \sum_n \bar{\pi}(n) \bar{F}(n) - \mathcal{F} \right| \le \sum_n \bar{\pi}(n) G(n).$$

A Recurrence Relation for the Bias Terms

- We will find variables $c(n,u,v,w)$ for which

$$D_u^{t+1}(n) = F(n+u) - F(n) + \sum_{v,w} c(n,u,v,w) D_v^t(n+w). \qquad (*)$$

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- Purpose of $c(n,u,v,w)$:

Lemma

If $A_u : S \to [0,\infty)$ and $B_u : S \to [0,\infty)$, $u \in N$ satisfy

$$F(n+u) - F(n) + \sum_{v,w} \max\{-c(\cdot)A_v(n+w), c(\cdot)B_v(n+w)\} \leq B_u(n),$$
$$F(n) - F(n+u) + \sum_{v,w} \max\{-c(\cdot)B_v(n+w), c(\cdot)A_v(n+w)\} \leq A_u(n),$$

for all $n \in S$ and $(*)$ is satisfied, then

$$-A_u(n) \leq D_u^t(n) \leq B_u(n),$$

for all $u \in N$, $n \in S$ and $t \geq 0$.

A Recurrence Relation for the Bias Terms (cont'd)

Lemma

We can formulate a set of linear constraints on $c(n, u, v, w)$ that ensure

$$D_u^{t+1}(n) = F(n+u) - F(n) + \sum_{v,w} c(n, u, v, w) D_v^t(n+w).$$

- Linear constraints in form of a flow problem
- $D_w^t(n) = F^t(n+w) - F^t(n)$
- $D_u^{t+1}(n) = F(n+u) - F(n) + \sum_{v,w} c(n, u, v, w) \left[F^t(n+w+v) - F^t(n+w)\right]$
- $F^{t+1}(n) = F(n) + \sum_u p_u(n) F^t(n+u)$
- $D_u^{t+1}(n) = F(n+u) - F(n) + \sum_v p_u(n) F^t(n+u+v) - \sum_w p_w(n) F^t(n+w)$

Markov Reward Error Bound

$$F^t(n) = F(n) + \sum_u p_u(n) F^{t-1}(n+u),$$
$$D_u^t(n) = F^t(n+u) - F^t(n).$$

Theorem (van Dijk)

Let $\bar{F} : S \to [0, \infty)$ and $G : S \to [0, \infty)$ satisfy

$$\left| \bar{F}(n) - F(n) + \sum_u \left[\bar{p}_u(n) - p_u(n) \right] D_u^t(n) \right| \leq G(n)$$

for all $n \in S$ and $t \geq 0$. Then

$$\left| \sum_n \bar{\pi}(n) \bar{F}(n) - \mathcal{F} \right| \leq \sum_n \bar{\pi}(n) G(n).$$

Let $q_u(n) = \bar{p}_u(n) - p_u(n)$.

A Finite Bilinear Program

- Upper bound $\mathcal{F}$ by

$$
\begin{aligned}
\min\ & \sum_{n \in S} \left[\bar{F}(n) + G(n)\right] \bar{\pi}(n), \\
\text{subject to } & \bar{F}(n) - F(n) + \sum_{u} \max\left\{q_u(n)B_u(n), -q_u(n)A_u(n)\right\} \leq G(n), \\
& F(n) - \bar{F}(n) + \sum_{u} \max\left\{q_u(n)A_u(n), -q_u(n)B_u(n)\right\} \leq G(n), \\
& F(n+u) - F(n) + \sum_{v,w} \max\{-c(\cdot)A_v(n+w), c(\cdot)B_v(n+w)\} \leq B_u(n), \\
& F(n) - F(n+u) + \sum_{v,w} \max\{-c(\cdot)B_v(n+w), c(\cdot)A_v(n+w)\} \leq A_u(n), \\
& \text{Constraints (*), linear in } c(\cdot), \\
& A_u(n) \geq 0, B_u(n) \geq 0, \bar{F}(n) \geq 0, G(n) \geq 0, \quad \text{for } n \in S, u \in N.
\end{aligned}
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\end{aligned}
$$

- Reduces to a **finite number of constraints and variables** if $\bar{F}$, A_u, B_u, G and c are constrained to be component-wise linear.

Software Implementation

Aim

General purpose software implementation in which transition probabilities and reward function are only input parameters.

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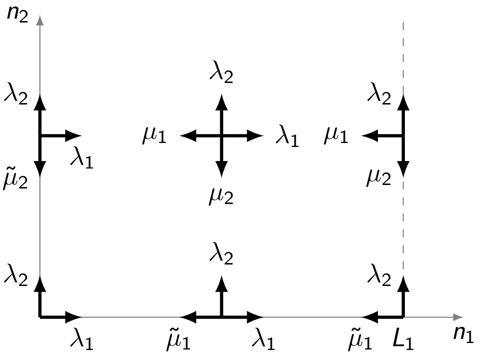
General purpose software implementation in which transition probabilities and reward function are only input parameters.

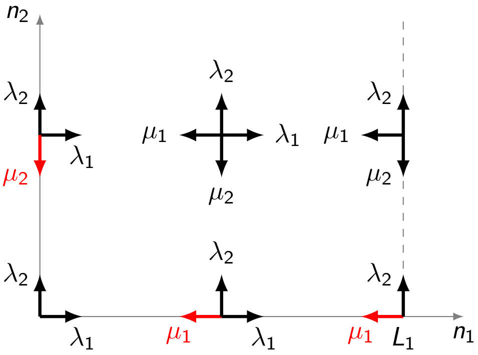
- Pyomo: a Python-based optimization modelling framework.
- Current implementation:
  - Additional input: perturbed random walk and its stationary distribution $\bar{\pi}$.
  - Find a feasible (not necessarily optimal) solution by first solving for (any) $c(n, u, v, w)$ and then the remaining linear program.

Example 1: Coupled Queue with a Finite Buffer



$\lambda_1 = \lambda_2 = 0.15$, $\mu_1 = \mu_2 = 0.2$, $\tilde{\mu}_1 = \tilde{\mu}_2 = 0.25$.

Example 1: Perturbed Random Walk

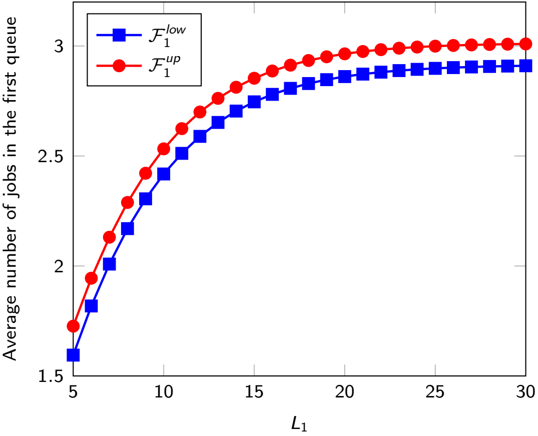


$\lambda_1 = \lambda_2 = 0.15$, $\mu_1 = \mu_2 = 0.2$, $\tilde{\mu}_1 = \tilde{\mu}_2 = 0.25$.

$\bar{\pi}(n) = \alpha(\frac{\lambda_1}{\mu_1})^{n_1}(\frac{\lambda_2}{\mu_2})^{n_2}$, where α is the normalization constant.

Example 1: Result

$F_1 = n_1$: average number of customers in the first queue



Perturbations for Two-dimensional Random Walks

Geometric product form random walks

- Geometric product form measure: $m(i,j) = \rho^i \sigma^j$, $(\rho, \sigma) \in (0,1)^2$.

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- Balance equation in interior: $m(i,j) = \sum_{s=-1}^{1} \sum_{t=-1}^{1} m(i-s, j-t) p_{s,t}$.
- Balance equations

$$\rho\sigma \left(1 - \sum_{s=-1}^{1} \sum_{t=-1}^{1} \rho^{-s} \sigma^{-t} p_{s,t}\right) = 0 \quad ,$$

Geometric product form random walks

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- Balance equations induce algebraic curves Q

$$Q = \left\{ (\rho, \sigma) \;\middle|\; \rho\sigma \left(1 - \sum_{s=-1}^{1} \sum_{t=-1}^{1} \rho^{-s} \sigma^{-t} p_{s,t} \right) = 0 \right\},$$

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- Balance equations induce algebraic curves Q, H and V,

$$Q = \left\{ (\rho,\sigma) \;\middle|\; \rho\sigma \left(1 - \sum_{s=-1}^{1} \sum_{t=-1}^{1} \rho^{-s} \sigma^{-t} p_{s,t} \right) = 0 \right\},$$

$$H = \left\{ (\rho,\sigma) \;\middle|\; \rho \left(1 - \sum_{s=-1}^{1} \rho^{-s} \sigma p_{s,-1} - \sum_{s=-1}^{1} \rho^{-s} h_s \right) = 0 \right\},$$

$$V = \left\{ (\rho,\sigma) \;\middle|\; \sigma \left(1 - \sum_{t=-1}^{1} \rho \sigma^{-t} p_{-1,t} - \sum_{t=-1}^{1} \sigma^{-t} v_t \right) = 0 \right\}.$$

Geometric product form random walks

- Geometric product form measure: $m(i,j) = \rho^i \sigma^j$, $(\rho, \sigma) \in (0,1)^2$.
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- Balance equations induce algebraic curves Q, H and V,

$$Q = \left\{ (\rho, \sigma) \;\middle|\; \rho\sigma \left(1 - \sum_{s=-1}^{1} \sum_{t=-1}^{1} \rho^{-s} \sigma^{-t} p_{s,t} \right) = 0 \right\},$$

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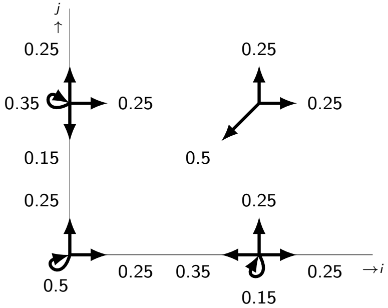
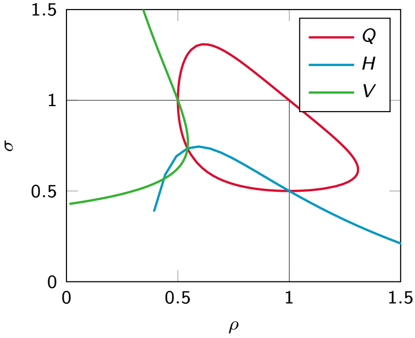
$$V = \left\{ (\rho, \sigma) \;\middle|\; \sigma \left(1 - \sum_{t=-1}^{1} \rho \sigma^{-t} p_{-1,t} - \sum_{t=-1}^{1} \sigma^{-t} v_t \right) = 0 \right\}.$$

- $m(i,j) = \rho^i \sigma^j$ is invariant measure iff $(\rho, \sigma) \in Q \cap H \cap V$.

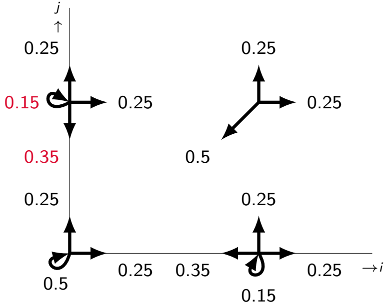
Product-form random walks (cont'd)

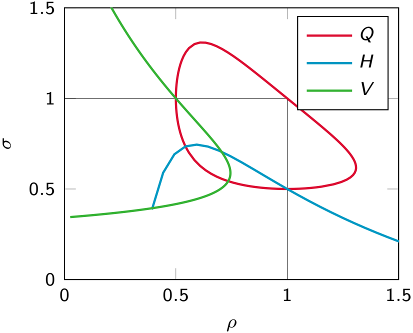
- $m(i,j) = \rho^i \sigma^j$ is invariant measure iff $(\rho, \sigma) \in Q \cap H \cap V$.
- Example of a product-form random walk:





- Example continued: Not a product form for other boundary transition rates





- Characterization of product-form queueing networks
  - van Dijk "Queueing Networks and Product Forms: a system's approach", 1993
  - Latouche & Miyazawa, QUESTA, 2014

Extending the class of ‘tractable’ random walks

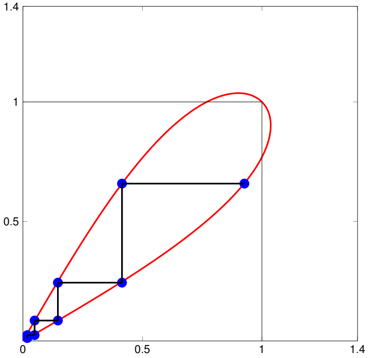
- Very limited number of random walks have a product-form invariant measure
- Goal: extend class of ‘tractable’ performance measures

Consider $m(i,j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j$

- Sum of geometric terms induced by Γ
- Assumptions:
  - $|\Gamma| < \infty$,
  - $\Gamma \subset (0,1)^2$,

Compensation approach

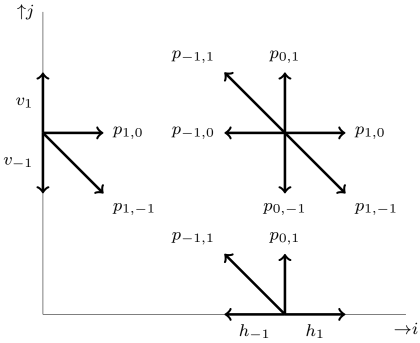
- Countably many terms with pairwise-coupled structure



Problem statement

- Sum of geometric terms: $m(i,j) = \sum_{(\rho,\sigma)\in\Gamma} \alpha(\rho,\sigma)\rho^i\sigma^j$
- When does a random walk have an invariant measure that can be expressed as a sum of geometric terms?
  - Which elements (ρ,σ) can occur in Γ?
  - Does Γ need to have a specific structure?
  - What are the values of $\alpha(\rho,\sigma)$?
- Can we construct a random walk that has a prespecified invariant measure that is a sum of geometric terms?

Example 2



↑j
v_1
v_{-1}
$p_{1,0}$
$p_{1,-1}$
$p_{-1,1}$
$p_{0,1}$
$p_{-1,0}$
$p_{1,0}$
$p_{0,-1}$
$p_{1,-1}$
$p_{-1,1}$
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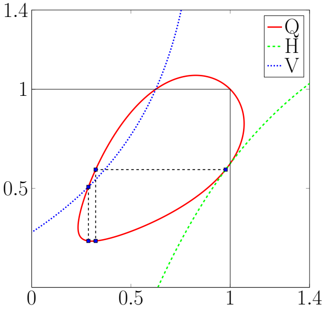
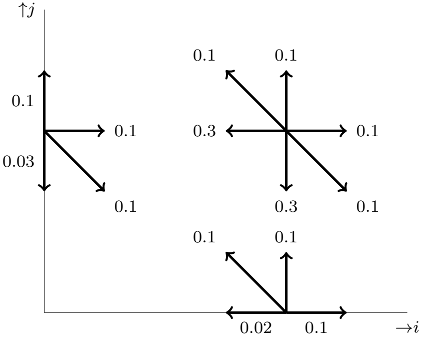
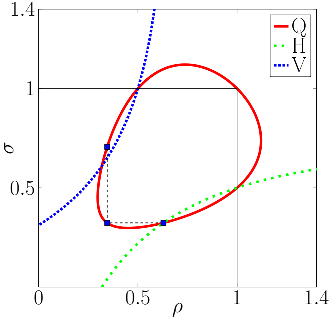
h_{-1}
h_1
→i



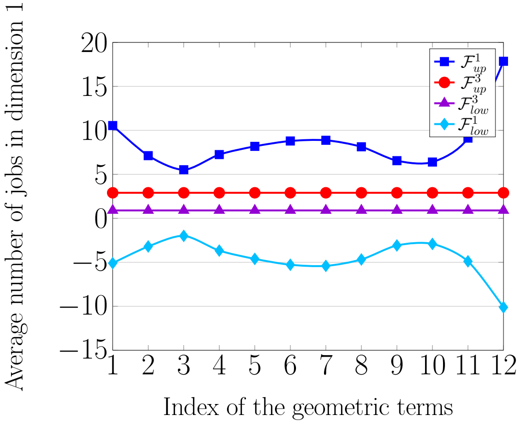
1.4
1
0.5
0
0.5
1
1.4
Q
H
V

Example 3





Example 3 (cont'd)



Future Work

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Thank you for your attention.